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Local State Probabilities for Solvable Restricted Solid-on-Solid Models: A_n, D_n, $D_n^{(1)}$, and $A_n^{(1)}$

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The local state probabilities (LSPs) are exactly computed for four hierarchies of solvable lattice models. They are restricted solid-on-solid (RSOS) models whose local states and their adjacent conditions are specified by Dinkin diagrams of types A_n, D_n, $D_n^{(1)}$, and $A_n^{(1)}$. The LSPs are expressed in terms of modular functions characterized by branching identities among the theta functions. Their automorphic properties are used to study the critical behaviors. Some fine structures are found in the spectrum of the critical exponents.

KEY WORDS: Statistical mechanics; solid-on-solid models; star-triangle relation; local state probability; modular function; branching coefficient.

1. STAR-TRIANGLE RELATIONS

1.1 Introduction

From the time Rogers–Ramanujan identities emerged in the analysis of Baxter's hard hexagon model,[1] crucial interplay has been seen between combinatorial aspects of modular functions and exactly solvable models in two-dimensional statistical mechanics. In these studies the physical quantity of central importance is a one-point function called the local state probability (LSP). It is by definition the probability $P(\lambda)$ that a lattice site assumes a given state λ. The computation of the LSPs by the corner transfer matrix method[2] amounts to evaluating combinatorial q-series in terms of modular functions. Such a program was first executed for a

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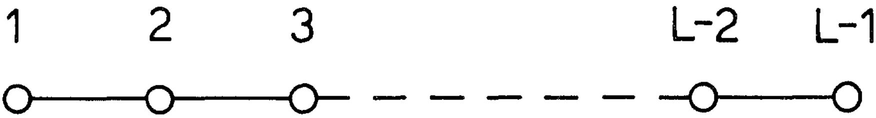

Fig. 1. Diagram for the restricted 8VSOS model. Each node corresponds to a local state. A pair of states is allowed to occupy adjacent lattice sites if the corresponding nodes are connected by a bond.

hierarchy of solvable models by Andrews *et al.*,[3] which they called restricted eight-vertex solid-on-solid (8VSOS) models. The models are labeled by an integer L ($\geqslant 4$) (r in their notation) and contain the hard hexagon model as the case $L=5$. An intriguing feature of the 8VSOS hierarchy is that[4] the LSPs exhibit the critical behavior exactly realizing the anomalous dimensions in minimal conformal field theory (CFT) by Belavin *et al.*[5] Now these results have been extended to a variety of solvable models,[6–11] bringing to light some intrinsic relations between solvable lattice models and CFTs.

In this paper we present yet further extensions of the 8VSOS models, evaluate their LSPs, and study the critical behaviors. In order to describe the extensions, let us recall the 8VSOS model where the state variable λ_i is assigned on site i of the square lattice and takes integer values $1 \leqslant \lambda_i \leqslant L-1$ with the restriction that neighboring states must differ by one. These conditions are described in Fig. 1. In Fig. 1 each node corresponds to a local state. Two states can occupy neighboring lattice sites if the corresponding nodes are connected by a bond in the diagram. Now consider Figs. 2–4. Pasquier[11] and Kuniba *et al.*[9,12] have obtained an elliptic solution to the star-triangle relation for the models corresponding to Figs. 2 and 3, respectively. The model associated with Fig. 4 is new. As in

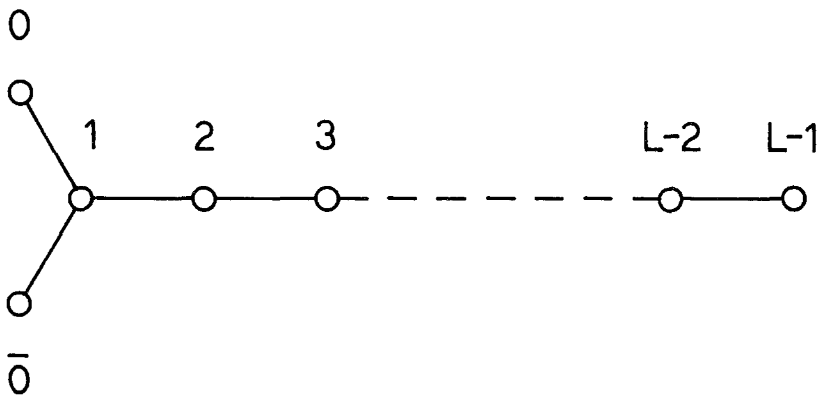

Fig. 2. Diagram for D_{L+1} model.

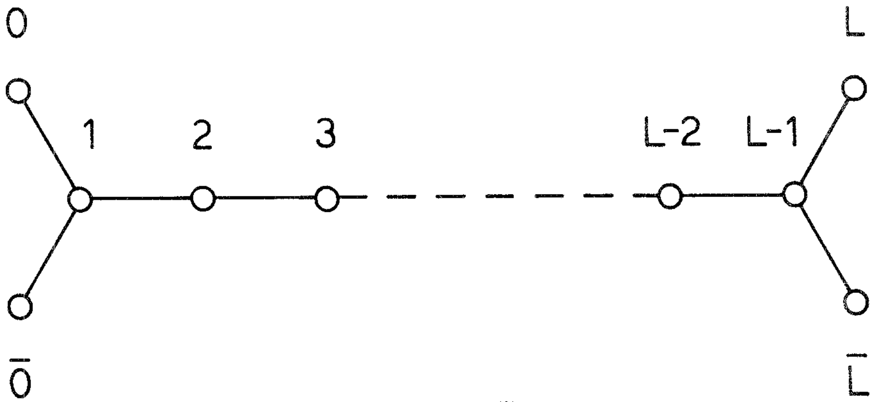

Fig. 3. Diagram for $D_{L+2}^{(1)}$ model.

the 8VSOS model, these models have four regimes (I–IV) of distinct physical behaviors. We obtain the LSPs in all regimes for these sequences of models, including the preliminary regime III results reported in refs. 9 and 11. It has been pointed out by Pasquier[13] that Figs. 1 and 2 are the Dynkin diagrams for the classical Lie algebras A_{L-1} and D_{L+1}, respectively. In this picture the diagrams in Figs. 3 and 4 correspond to those for affine Lie algebras $D_{L+2}^{(1)}$ and $A_{L-1}^{(1)}$, respectively. For convenience we call the models specified by Figs. 1–4 and the elliptic solutions to the STR as A_{L-1}, D_{L+1}, $D_{L+2}^{(1)}$, and $A_{L-1}^{(1)}$ models, respectively. We note that this viewpoint is different from the recent work by Jimbo *et al.*,[14] where the local states take their values in dominant integral weights of affine Lie algebras with fixed level.

The principal feature of our model $D_{L+2}^{(1)}$ is that the essential part of the LSPs in regime III (generating function for the eigenvalue spectrum of corner transfer matrices) is expressed by functions of the form [see (A.21) in the Appendix]:

$$\eta(\tau)^{-1} \sum_{\lambda \in Z} \left(q^{[2\lambda L(L-1)+rL-a(L-1)]^2/4L(L-1)} + q^{[2\lambda L(L-1)+rL+a(L-1)]^2/4L(L-1)}\right) \tag{1.1}$$

$$r, a \in Z, \qquad 0 \leqslant r \leqslant L-1, \qquad 0 \leqslant a \leqslant L$$

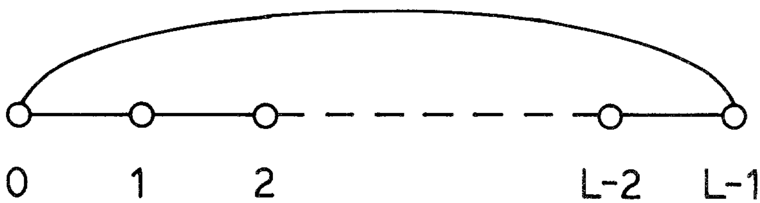

Fig. 4. Diagram for $A_{L-1}^{(1)}$ model.

while in the A_{L-1} model the corresponding quantity is

$$\eta(\tau)^{-1} \sum_{\lambda \in Z} (q^{[2\lambda L(L-1)+rL-a(L-1)]^2/4L(L-1)} - q^{[2\lambda L(L-1)+rL+a(L-1)]^2/4L(L-1)}) \tag{1.2}$$

$$r, a \in Z, \qquad 0<r<L-1, \qquad 0<a<L$$

here $\eta(\tau)$ is Dedekind's eta function [see (1.7)]. The function (1.2) is the irreducible character of Virasoro algebra.[15] The LSPs for $D_{L/2+1}$ (L even) and $A^{(1)}_{L-1}$ models are expressed in terms of certain combinations of (1.1) and (1.2) (see Table V). This yields several fine structures in the critical behaviors.

The organisation of the paper is as follows.

In the remainder of this section, we recall the elementary facts about the star-triangle relations for a class of restricted solid-on-solid models and give the elliptic solutions for the present models.

In Section 2, we express the LSPs in terms of one-dimensional configuration sums $X_m(a, b, c)$ and present the results in the limit $m \to \infty$.

In Section 3, we study the one-dimensional configuration sums. We rewrite them in series involving Gaussian polynomials and identify them with modular functions (or branching coefficients) in the limit of m large. This yields the LSP results summarized in Section 2.3.

In Section 4, we investigate the critical behaviors of the LSPs by utilizing the automorphic properties of the modular functions.

Section 5 gives a summary and discussions.

The appendix gives the definitions and the basic properties of the theta functions and branching coefficients used in the main text.

Throughout the paper we use the following notations:

$$E(z, q) = \prod_{n=1}^{\infty} (1 - zq^{n-1})(1 - z^{-1}q^n)(1 - q^n) \tag{1.3}$$

$$\theta_1(u, q^2) = 2\,|q|^{1/4} \sin u \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2u + q^{4n})(1 - q^{2n}) \tag{1.4}$$

$$\theta_4(u, q^2) = \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2u + q^{4n-2})(1 - q^{2n}) \tag{1.5}$$

$$\phi(q) = \prod_{n=1}^{\infty} (1 - q^n) \tag{1.6}$$

$$\eta(\tau) = q^{1/24}\, \phi(q), \qquad q = e^{2\pi i\tau} \tag{1.7}$$

(in Sections 2–4, we fix this relation between q and τ);

$$\begin{aligned} \varepsilon_k^j &= 1/2 \qquad \text{if} \quad k \equiv 0 \bmod j \\ &= 1 \qquad\quad \text{otherwise.} \end{aligned} \tag{1.8}$$

In particular, $\varepsilon_0^\infty = 1/2$ and $\varepsilon_j^\infty = 1$ for $j \neq 0$.

At every stage we shall rephrase the results in ref. 3 or corresponding $N=1$ results in refs. 6–8 in suitable forms for comparison and clarify the significant difference of the models from each other.

1.2. Restricted Solid-on-Solid Models

Before going into our specific models A_{L-1}, D_{L+1}, $D_{L+2}^{(1)}$, and $A_{L-1}^{(1)}$, we briefly summarize the basic facts about a class of restricted solid-on-solid (RSOS) models. Consider a two-dimensional square lattice with a fluctuation variable λ_i associated to each site i. We shall call the λ_i a state and assume that $\lambda_i \in S$ with S being a finite set of the states. Let s denote the number of elements in S $(s>1)$ and consider an s by s matrix C satisfying the following conditions:

(i) $C_{\lambda,\lambda'} = C_{\lambda',\lambda} = 0$ or 1 (1.9a)

(ii) $C_{\lambda,\lambda} = 0$ (1.9b)

(iii) For each $\lambda \in S$, there exist $\lambda' \in S$ such that $C_{\lambda',\lambda} = 1$ (1.9c)

For such choice of C, we impose a restriction that two states λ and λ' can occupy the neighboring lattice sites if and only if $C_{\lambda,\lambda'} = 1$. We shall call such a pair of the states (λ, λ') admissible. These conditions are conveniently expressed by a connected graph as in Figs. 1–4, where each node corresponds to a state and the admissibility specified by a bond. Let λ_i, λ_j, λ_k, and λ_l be the four states assigned on the lattice sites i, j, k, and l surrounding a face. We assume that an elementary interaction is given by a Boltzmann weight $W(\lambda_i, \lambda_j, \lambda_k, \lambda_l)$ attached to a state configuration around a face depicted in Fig. 5. We also assume that $W(\lambda_i, \lambda_j, \lambda_k, \lambda_l)$ is zero unless the four pairs (λ_i, λ_j), (λ_j, λ_k), (λ_k, λ_l), and (λ_l, λ_i) are admissible. RSOS models our of concern are the interaction-round-face (IRF) models in the sense of Baxter[2] with the above conditions on the state variables and the Boltzmann weights. We remark that these RSOS models are "nonoriented" in the sense that $C_{\lambda,\lambda'} = C_{\lambda',\lambda}$.

Now we proceed to the description of the star-triangle relation (STR), which assures the solvability of the models. We introduce a spectral parameter u and assume that the Boltzmann weights are functions of u. The

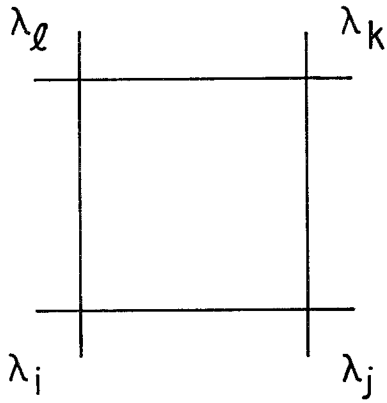

Fig. 5. A state configuration $(\lambda_i, \lambda_j, \lambda_k, \lambda_l)$ around a face, where the sites i, j, k, and l are ordered anticlockwise from the southwest corner.

STR is the following system of functional equations for the Boltzmann weights:

$$\sum_g W(a, b, g, f \mid u)\, W(f, g, d, e \mid u+v)\, W(g, b, c, d \mid v)$$

$$= \sum_g W(f, a, g, e \mid v)\, W(a, b, c, g \mid u+v)\, W(g, c, d, e \mid u) \tag{1.10}$$

where the sum on the lhs (resp. rhs) is taken over $g \in S$ such that the pairs (g, b), (g, d), and (g, f) [resp. (g, a), (g, c), and (g, e)] are admissible. It is known that the STR (1.10) is stated also by using the face operators, which we now explain. Consider the one-dimensional configuration of the states $\{\lambda_j\}_{j=1}^m$ with m being a positive integer sufficiently large. By relabeling we may assume that $S = \{1, 2,..., s\}$. Let V be the subspace of $C^s \otimes \cdots \otimes C^s$ (m-fold tensor product) spanned by the vectors $e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_m}$ such that $(\lambda_j, \lambda_{j+1})$ is admissible for $1 \leqslant j \leqslant m-1$. Here e_j stands for the standard basis of C^s. Define the jth face operator $R_j(u)$ $(2 \leqslant j \leqslant m-1)$ acting on V by

$$R_j(u)\, e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_m}$$

$$= \sum_{\lambda'_j \in S} W(\lambda_j, \lambda_{j+1}, \lambda'_j, \lambda_{j-1} \mid u)\, e_{\lambda_1} \otimes \cdots \otimes e_{\lambda'_j} \otimes \cdots e_{\lambda_m} \tag{1.11}$$

In terms of the operator $R_j(u)$, the STR (1.10) is rephrased as

$$R_i(u)\, R_{i+1}(u+v)\, R_i(v) = R_{i+1}(v)\, R_i(u+v)\, R_{i+1}(u) \tag{1.12}$$

Under the assumptions (i) and (ii) in the sequel, the STR (1.10) or (1.12) considerably simplifies.

(i) Let the face operator $R_j(u)$ be of the Temperly–Lieb type.[16,17] By this we mean the following form of $R_j(u)$:

$$R_j(u) = \rho(u)[I + y(u)\, U_j] \tag{1.13a}$$

$$\rho(u) = \frac{\sin(\mu - u)}{\sin \mu}, \qquad y(u) = \frac{\sin u}{\sin(\mu - u)} \tag{1.13b}$$

where μ is a parameter such that $\sin \mu \neq 0$ and I is an identity operator in End V. The U_j $(2 \leqslant j \leqslant m-1)$ in (1.13a) is a u-independent operator in End V that obeys the Temperly–Lieb algebra:

$$U_j U_{j\pm 1} U_j = U_j, \qquad U_i U_j = U_j U_i \qquad \text{for} \quad |i-j| > 1 \tag{1.14a}$$

$$U_j^2 = q^{1/2} U_j, \qquad q^{1/2} = 2 \cos \mu \tag{1.14b}$$

It can be directly checked that the the STR (1.12) is assured by the relations (1.14) among the Temperly–Lieb operators U_j $(2 \leqslant j \leqslant m-1)$.

(ii) To each state $\lambda \in S$, assign a complex number $g_\lambda \neq 0$ and assume the following form for the (λ, λ') element of the operator U_j:

$$\begin{aligned}(U_j)_{\lambda,\lambda'} = {} & \delta(\lambda_1, \lambda'_1) \cdots \delta(\lambda_{j-1}, \lambda'_{j-1}) \\ & \times \delta(\lambda_{j-1}, \lambda_{j+1}) \frac{g_{\lambda_j} g_{\lambda'_j}}{g_{\lambda_{j-1}} g_{\lambda_{j+1}}} \\ & \times \delta(\lambda_{j+1}, \lambda'_{j+1}) \cdots \delta(\lambda_m, \lambda'_m)\end{aligned} \tag{1.15}$$

Among the relations in the Temperley–Lieb algebra, (1.14a) is automatically satisfied by this form of U_j. On the other hand, the condition (1.14b) is reduced to linear equations for the parameters g_λ^2:

$$\sum_{\substack{\lambda' \\ (\lambda,\lambda')\text{:admissible}}} g_{\lambda'}^2 = q^{1/2} g_\lambda^2, \tag{1.16a}$$

In terms of the vector $\mathbf{h} \in C^s$, $(\mathbf{h})_\lambda = g_\lambda^2$, (1.16a) is written as

$$C\mathbf{h} = q^{1/2}\mathbf{h} \tag{1.16b}$$

where C is the s by s matrix introduced in (1.9). Note that the ansatz

(1.13)–(1.15) for the face operator $R_j(u)$ amounts to the following form for the Boltzmann weight (1.11):

$$W(\lambda_i, \lambda_j, \lambda_k, \lambda_l \mid u) = \frac{\sin(\mu - u)}{\sin \mu} \delta(\lambda_i, \lambda_k) + \frac{g_{\lambda_i} g_{\lambda_k}}{g_{\lambda_j} g_{\lambda_l}} \frac{\sin u}{\sin \mu} \delta(\lambda_j, \lambda_l) \tag{1.17}$$

We see that (1.17) satisfies the following properties.

Reflection symmetry:

$$\begin{aligned} W(\lambda_i, \lambda_j, \lambda_k, \lambda_l \mid u) &= W(\lambda_k, \lambda_j, \lambda_i, \lambda_l \mid u) \\ &= W(\lambda_i, \lambda_l, \lambda_k, \lambda_j \mid u) \end{aligned} \tag{1.18}$$

Crossing symmetry

$$W(\lambda_i, \lambda_j, \lambda_k, \lambda_l \mid u) = \frac{g_{\lambda_i} g_{\lambda_k}}{g_{\lambda_j} g_{\lambda_l}} W(\lambda_l, \lambda_i, \lambda_j, \lambda_k \mid \mu - u) \tag{1.19}$$

The Temperly–Lieb operator for the restricted 8VSOS models was first extracted in a form (1.15) in ref. 18.

Thus, a class of trigonometric solutions is obtained for the STR by solving the eigenvalue problem (1.16). This much is true for arbitrary choice of the matrix C in (1.9) (in fact, even for the case $C_{\lambda,\lambda} = 1$). Let us see what happens if we further impose a kind of physical condition:

$$\mu,\ g_\lambda(\neq 0) \in R \qquad \text{for all} \quad \lambda \in S \tag{1.20}$$

From (1.14b) and (1.16a) we deduce $0 < q^{1/2} < 2$ ($q^{1/2} = 2$ is forbidden by the assumption $\sin \mu \neq 0$). For such cases a complete list for the matrix $2 - C$ satisfying (1.9) and (1.16b) is available as the classical Cartan matrix of types A, D, and E. These cases are considered by Pasquier.[13] On the other hand, the formal choice $q^{1/2} = 2$ ($\sin \mu = 0$) in (1.17) does not lead to nontrivial solutions for the STR, although such a matrix $2 - C$ is classified as the generalized Cartan matrix[19] for affine Lie algebras $A^{(1)}$, $D^{(1)}$, and $E^{(1)}$. In both cases the $\mathbf{h}$ in (1.16b) becomes the Perron–Frobenius vector for the matrix C [see also (4.8)]. We shall encounter the distinct nature of the cases $0 < q < 4$ (Figs. 1 and 2) and $q = 4$ (Figs. 3 and 4) in the rest of the paper in various aspects: STR, LSPs, critical behaviors, etc.

1.3. Elliptic Solutions to STR

Let us return to our models A_{L-1}, D_{L+1}, $D^{(1)}_{L+2}$, and $A^{(1)}_{L-1}$, whose matrices C are diagrammatically shown in Figs. 1–4. We set $\lambda_i \in S$ with

$$S = \{1, 2, ..., L-1\} \qquad \text{for model } A_{L-1}, \quad L \geqslant 4 \tag{1.21a}$$

$$= \{0, \bar{0}, 1, 2, ..., L-1\} \qquad \text{for model } D_{L+1}, \quad L \geqslant 3 \tag{1.21b}$$

$$= \{0, \bar{0}, 1, 2, ..., L-1, L, \bar{L}\} \qquad \text{for model } D^{(1)}_{L+2}, \quad L \geqslant 3 \tag{1.21c}$$

$$= \{0, 1, 2, ..., L-1\} \qquad \text{for model } A^{(1)}_{L-1}, \quad L \geqslant 3 \tag{1.21d}$$

We remark that the admissibility condition in the $D^{(1)}_5$ model is equivalent to that in the odd-height sector of the fusion model[6–8] with $(L, N) = (6, 2)$, i.e., $l_i - l_j = 0, \pm 2, l_i + l_j = 4, 6, 8$. This can be seen by relabeling the state variables in the $D^{(1)}_5$ model as in Fig. 6. Originally this model was solved in ref. 20 and extended to the present $D^{(1)}_n$ model for arbitrary n ($\geqslant 5$) in refs. 9 and 12.

Below we present the elliptic parametrization of the Boltzmann weights satisfying the STR. For all the models A_{L-1}, D_{L+1}, $D^{(1)}_{L+2}$, and $A^{(1)}_{L-1}$, they enjoy the reflection symmetry (1.18) and the crossing symmetry (1.19) with $\mu = -1$ and the g_λ specified as follows (the parameter u has been rescaled so as to make $\mu = -1$ for all the models):

(a) A_{L-1} model

$$g_\lambda = \varepsilon_\lambda [\theta_1(\pi\lambda/L, p)]^{1/2} \tag{1.22a}$$

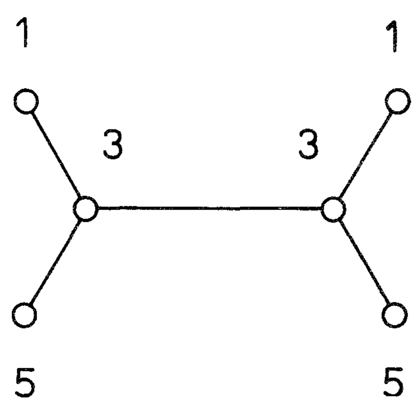

Fig. 6. The relabeling of the state variables $\{0, \bar{0}, 1, 2, 3, \bar{3}\}$ in the $D^{(1)}_5$ model, showing the equivalence of admissibility with that in the odd-height sector of the fusion model[6–8] with $(L, N) = (6, 2)$.

(b) D_{L+1} model

$$\begin{aligned} g_\lambda &= \varepsilon_\lambda[\varepsilon_\lambda^\infty \theta_1(\pi(L+\lambda)/2L, p)]^{1/2} \qquad \text{for} \quad \lambda \neq \bar{0} \\ g_{\bar{0}} &= g_0 \end{aligned} \tag{1.22b}$$

(c) $D_{L+2}^{(1)}$ model

$$\begin{aligned} g_\lambda &= \varepsilon_\lambda[\varepsilon_\lambda^L \theta_4(\pi\lambda/L, p)]^{1/2} \qquad \text{for} \quad \lambda \neq \bar{0}, \bar{L} \\ g_{\bar{0}} &= g_0, \qquad g_{\bar{L}} = g_L \end{aligned} \tag{1.22c}$$

(d) $A_{L-1}^{(1)}$ model

$$g_\lambda = \varepsilon_\lambda[\theta_4(\pi\lambda/L, p)]^{1/2} \tag{1.22d}$$

In the above, θ_1 and θ_4 are the elliptic theta functions defined in (1.4), (1.5) with "nome" p ($|p|<1$), the symbols ε_λ^L and $\varepsilon_\lambda^\infty$ are specified by (1.8), and $\varepsilon_\lambda = \pm 1$, $\varepsilon_\lambda \varepsilon_{\lambda+1} = (-)^\lambda$. In the trigonometric limit $p \to 0$, the relation (1.16) holds with the following values of $q^{1/2}$:

$$q^{1/2} = 2\cos(\pi/L) \qquad \text{for} \quad A_{L-1} \text{ model} \tag{1.23a}$$

$$= 2\cos(\pi/2L) \qquad \text{for} \quad D_{L+1} \text{ model} \tag{1.23b}$$

$$= 2 \qquad \text{for} \quad D_{L+2}^{(1)} \text{ and } A_{L-1}^{(1)} \text{ models} \tag{1.23c}$$

We remark that in the A_{L-1} and D_{L+1} models, the Boltzmann weights themselves reduce to the trigonometric ones given in Section 1.2 in the limit $p \to 0$.

(a) A_{L-1} model:

$$\begin{aligned} W(\lambda, \lambda+1, \lambda, \lambda-1) &= \frac{H(1+u)}{H(1)}, \qquad 2 \leqslant \lambda \leqslant L-2 \\ W(\lambda+1, \lambda, \lambda-1, \lambda) &= \frac{[H(\lambda+1)\,H(\lambda-1)]^{1/2}}{H(\lambda)} \frac{H(u)}{H(1)}, \quad 2 \leqslant \lambda \leqslant L-2 \\ W(\lambda \pm 1, \lambda, \lambda \pm 1, \lambda) &= \frac{H(\lambda \mp u)}{H(\lambda)}, \qquad 1 \leqslant \lambda, \lambda \pm 1 \leqslant L-1 \end{aligned} \tag{1.24}$$

where $H(u) = \theta_1(\pi u/L, p)$. The parametrization has the property

$$W(a, b, c, d \mid u) = W(L-a, L-b, L-c, L-d \mid u) \tag{1.25}$$

(b) D_{L+1} model

$$W(\lambda, \lambda+1, \lambda, \lambda-1) = \frac{H(1+u)}{H(1)}, \qquad 1 \leqslant \lambda \leqslant L-2$$

$$W(\lambda+1, \lambda, \lambda-1, \lambda) = \frac{[\varepsilon_{\lambda-1}^{\infty} H(L+\lambda+1)\, H(L+\lambda-1)]^{1/2}}{H(L+\lambda)} \times \frac{H(u)}{H(1)}, \qquad 1 \leqslant \lambda \leqslant L-2$$

$$W(2, 1, \bar{0}, 1) = W(2, 1, 0, 1)$$

$$W(\lambda \pm 1, \lambda, \lambda \pm 1, \lambda) = \frac{H(L+\lambda \mp u)}{H(L+\lambda)}, \qquad 1 \leqslant \lambda, \lambda \pm 1 \leqslant L-1 \quad (1.26)$$

$$W(1, \bar{0}, 1, 0) = \frac{H(L-u)}{H(L)} + \frac{H(L+1)\, H(u)}{H(L)\, H(1)}$$

$$W(1, 0, 1, 0) = \frac{H(L-u)}{H(L)} - \frac{H(L+1)\, H(u)}{H(L)\, H(1)}$$

$$W(0, 1, \bar{0}, 1) = \frac{1}{2}\left(\frac{H(L+1+u)}{H(L+1)} - \frac{H(1+u)}{H(1)}\right)$$

$$W(0, 1, 0, 1) = \frac{1}{2}\left(\frac{H(L+1+u)}{H(L+1)} + \frac{H(1+u)}{H(1)}\right)$$

where $H(u) = \theta_1(\pi u/2L, p)$. The parametrization has the property

$$W(a, b, c, d \mid u) = W(\bar{a}, \bar{b}, \bar{c}, \bar{d} \mid u) \quad (1.27)$$

where $\bar{\lambda}$ is defined by

$$\bar{\lambda} = \lambda \quad \text{if} \quad \lambda \neq 0, \qquad \overline{(0)} = \bar{0}, \qquad \overline{(\bar{0})} = 0 \quad (1.28)$$

(c) $D_{L+2}^{(1)}$ model:

$$W(\lambda, \lambda+1, \lambda, \lambda-1) = \frac{H(1+u)}{H(1)}, \qquad \lambda \neq 0, \bar{0}, L, \bar{L}$$

$$W(\lambda+1, \lambda, \lambda-1, \lambda) = \frac{[\varepsilon_{\lambda+1}^{L} \Theta(\lambda+1)\, \varepsilon_{\lambda-1}^{L} \Theta(\lambda-1)]^{1/2}}{\Theta(\lambda)} \times \frac{H(u)}{H(1)}, \qquad \lambda \neq 0, \bar{0}, L, \bar{L}$$

$$W(\lambda \pm 1, \lambda, \lambda \pm 1, \lambda) = \frac{\Theta(\lambda \mp u)}{\Theta(\lambda)}, \qquad \lambda, \lambda \pm 1 \neq 0, \bar{0}, L, \bar{L} \quad (1.29)$$

$$W(0,1,0,1)=W(L,L-1,L,L-1)=\frac{1}{2}\left(\frac{\Theta(1+u)}{\Theta(1)}+\frac{H(1+u)}{H(1)}\right)$$

$$W(0,1,\bar{0},1)=W(L,L-1,\bar{L},L-1)=\frac{1}{2}\left(\frac{\Theta(1+u)}{\Theta(1)}-\frac{H(1+u)}{H(1)}\right)$$

$$W(1,0,1,0)=W(L-1,L,L-1,L)=\frac{\Theta(u)}{\Theta(0)}-\frac{\Theta(1)\,H(u)}{\Theta(0)\,H(1)}$$

$$W(1,0,1,\bar{0})=W(L-1,L,L-1,\bar{L})=\frac{\Theta(u)}{\Theta(0)}+\frac{\Theta(1)\,H(u)}{\Theta(0)\,H(1)}$$

where $H(u)=\theta_1(\pi u/L, p)$ and $\Theta(u)=\theta_4(\pi u/L, p)$. The following two properties are valid.

Property (i):

$$W(a,b,c,d\,|\,u)=W(a^*,b^*,c^*,d^*\,|\,u) \tag{1.30a}$$

where λ^* is defined by

$$\lambda^*=L-\lambda \qquad \text{for} \quad \lambda\neq\bar{0},\bar{L}, \qquad (\bar{0})^*=\bar{L}, \qquad (\bar{L})^*=\bar{0} \tag{1.30b}$$

Property (ii):

$$W(a,b,c,d\,|\,u)=W(\bar{a},\bar{b},\bar{c},\bar{d}\,|\,u) \tag{1.31a}$$

where $\bar{\lambda}$ is defined by

$$\begin{gathered}\bar{\lambda}=\lambda \qquad \text{for} \quad \lambda\neq 0,\bar{0},L,\bar{L}\\ \overline{(L)}=\bar{L}, \qquad \overline{(\bar{L})}=L, \qquad \overline{(0)}=\bar{0}, \qquad \overline{(\bar{0})}=0\end{gathered} \tag{1.31b}$$

(d) $A^{(1)}_{L-1}$ model

$$\begin{gathered}W(\lambda,\lambda+1,\lambda,\lambda-1)=\frac{H(1+u)}{H(1)}, \qquad 1\leqslant\lambda\leqslant L-2\\ W(0,1,0,L-1)=W(L-1,0,L-1,L-2)=\frac{H(1+u)}{H(1)}\\ W(\lambda+1,\lambda,\lambda-1,\lambda)=\frac{[\Theta(\lambda+1)\,\Theta(\lambda-1)]^{1/2}}{\Theta(\lambda)}\frac{H(u)}{H(1)}, \quad 1\leqslant\lambda\leqslant L-2\\ W(1,0,L-1,0)=\frac{\Theta(1)\,H(u)}{\Theta(0)\,H(1)}\\ W(0,L-1,L-2,L-1)=\frac{[\Theta(0)\,\Theta(2)]^{1/2}}{\Theta(1)}\frac{H(u)}{H(1)}\end{gathered} \tag{1.32}$$

$$W(\lambda \pm 1, \lambda, \lambda \pm 1, \lambda) = \frac{\Theta(\lambda \mp u)}{\Theta(\lambda)}, \qquad 0 \leqslant \lambda, \lambda \pm 1 \leqslant L-1$$

$$W(L-1, 0, L-1, 0) = \frac{\Theta(u)}{\Theta(0)}$$

$$W(0, L-1, 0, L-1) = \frac{\Theta(1+u)}{\Theta(1)}$$

where $H(u) = \theta_1(\pi u/L, p)$ and $\Theta(u) = \theta_4(\pi u/L, p)$.

2. LOCAL STATE PROBABILITIES

We employ Baxter's corner transfer matrix method to compute the local state probabilities (LSPs) of our models. We refer to Appendix A in ref. 3 for the description of this method adapted to the present context.

2.1. Multiple Sum Expressions

There are four regimes (I–IV) for each model exhibiting distinct physical behaviors depending on the values of the parameters u and p. We specify them as follows.

(a) A_{L-1} model:

$$\begin{array}{lll} \text{Regime I:} & -1 < p < 0, & 0 < u < L/2 - 1 \\ \text{Regime II:} & 0 < p < 1, & 0 < u < L/2 - 1 \\ \text{Regime III:} & 0 < p < 1, & -1 < u < 0 \\ \text{Regime IV:} & -1 < p < 0, & -1 < u < 0 \end{array} \tag{2.1a}$$

(b) D_{L+1} model:

$$\begin{array}{lll} \text{Regime I:} & -1 < p < 0, & 0 < u < L - 1 \\ \text{Regime II:} & 0 < p < 1, & 0 < u < L - 1 \\ \text{Regime III:} & 0 < p < 1, & -1 < u < 0 \\ \text{Regime IV:} & -1 < p < 0, & -1 < u < 0 \end{array} \tag{2.1b}$$

(c) $D^{(1)}_{L+2}$ and $A^{(1)}_{L-1}$ models:

$$\begin{array}{lll} \text{Regime I:} & -1 < p^{1/2} < 0, & 0 < u < L/2 - 1 \\ \text{Regime II:} & 0 < p^{1/2} < 1, & 0 < u < L/2 - 1 \\ \text{Regime III:} & 0 < p^{1/2} < 1, & -1 < u < 0 \\ \text{Regime IV:} & -1 < p^{1/2} < 0, & -1 < u < 0 \end{array} \tag{2.1c}$$

Let $a, b, c \in S$ be three states in (1.21) such that the pair (b, c) is admissible. The LSP $P(a \mid \Lambda)$ is the probability that a state variable λ_1 takes a given state $\lambda_1 = a$ under the condition that those far from the site 1 are fixed to a certain background configuration Λ specified by (b, c). (See Section 2.2 for a precise description of the background configurations and the ground states.) Let m be an integer satisfying $m \geqslant 1$. By the method of corner transfer matrix the LSP is reduced to the $m \to \infty$ limit of the quantity $P_m(a \mid b, c)$:

$$P_m(a \mid b, c) = x^{\xi_a} u_a X_m(a, b, c : q^\sigma)/N_m(b, c) \tag{2.2a}$$

$$N_m(b, c) = \sum_{a \in S} x^{\xi_a} u_a X_m(a, b, c : q^\sigma) \tag{2.2b}$$

$$X_m(a, b, c : q) = \sum q^{S_m(\lambda_1, \dots, \lambda_{m+2})} \tag{2.2c}$$

$$S_m(\lambda_1, \dots, \lambda_{m+2}) = \sum_{j=1}^{m} jH(\lambda_j, \lambda_{j+1}, \lambda_{j+2}) \tag{2.2d}$$

Here the sum in (2.2c) extends over the state variables $\lambda_2, \lambda_3, \dots, \lambda_m$ under the condition that $\lambda_1 = a$, $\lambda_{m+1} = b$, $\lambda_{m+2} = c$ and the pair $(\lambda_j, \lambda_{j+1})$ is admissible for $1 \leqslant j \leqslant m$. The quantities x, q, σ, ξ_a, u_a are listed in Table I for the A_{L-1}, D_{L+1}, $D_{L+2}^{(1)}$, and $A_{L-1}^{(1)}$ models.

The weight function $H(a, b, c)$ in (2.2d) is defined for three states a, b, c such that the pairs (a, b) and (b, c) are admissible. Their explicit forms are given in (2.5)–(2.13c). We remark that the symmetries of the Boltzmann weights have been lost from the weight function $H(a, b, c)$. For example, from (2.6a), (2.6b) we see that $H(1, 0, 1) \neq H(1, \bar{0}, 1)$ for the D_{L+1} model in regimes II and III, while we have the $\lambda \leftrightarrow \bar{\lambda}$ symmetry (1.27) in the Boltzmann weights. In order to solve this apparent contradiction, let us describe some details of the prescription to derive the function $H(a, b, c)$.

In Section 1.3 the Boltzmann weights are given as the functions of the spectral parameter u and the elliptic nome p. Using the conjugate modulus identities listed in Eq. (3.3.6) of ref. 3, one can rewrite them in terms of the variables x defined in Table I and $w = x^u$ [the variable w should not be confused with the Boltzmann weight $W(a, b, c, d)$]. The first step to obtain the weight function $H(a, b, c)$ is to take the following limit of the Boltzmann weight:

$$U(a, c)_{b,d} = \lim_{\substack{x \to 0, u \to 0 \\ w = x^u : \text{fixed}}} W(b, c, d, a) \frac{G_b G_d}{G_a G_c} \Big/ F \tag{2.3}$$

where the quantities G_a and F are specified in Table II. One may regard

$U(a, c)_{b,d}$ as the (b, d) element of face transfer matrix (in the $x \to 0$ limit) $U(a, c)$ in the SW–NE direction. The size of $U(a, c)$ is the number of the state λ such that the pairs (λ, a) and (λ, c) are admissible. Direct calculation shows that all the matrices $U(a, c)$ are diagonal except for the one listed in Table III.

Table I. Parameters for A_{L-1}, D_{L+1}, $D^{(1)}_{L+2}$, and $A^{(1)}_{L-1}$ Models

	I	II	III	IV
A_{L-1} model[a]				
p	$-\exp(-\varepsilon/L)$	$\exp(-\varepsilon/L)$	$\exp(-\varepsilon/L)$	$-\exp(-\varepsilon/L)$
x	$\exp(-2\pi^2/\varepsilon)$	$\exp(-4\pi^2/\varepsilon)$	$\exp(-4\pi^2/\varepsilon)$	$\exp(-2\pi^2/\varepsilon)$
q	x^{L-2}	x^{L-2}	x^2	x^2
σ	$+1$	-1	$+1$	-1
u_a	$E(x^a, -x^{L/2})$	$E(x^a, x^L)$	$E(x^a, x^L)$	$E(x^a, -x^{L/2})$
ξ_a	$a(a-L+1)/2$	$a(a-L)/4$	0	$a/2$
D_{L+1} model[b]				
p	$-\exp(-\varepsilon/2L)$	$\exp(-\varepsilon/2L)$	$\exp(-\varepsilon/2L)$	$-\exp(-\varepsilon/2L)$
x	$\exp(-2\pi^2/\varepsilon)$	$\exp(-4\pi^2/\varepsilon)$	$\exp(-4\pi^2/\varepsilon)$	$\exp(-2\pi^2/\varepsilon)$
q	x^{2L-2}	x^{2L-2}	x^2	x^2
σ	$+1$	-1	$+1$	-1
u_a	$\varepsilon^{\infty}_a E(x^{a+L}, -x^L)$	$\varepsilon^{\infty}_a E(x^{a+L}, x^{2L})$	$\varepsilon^{\infty}_a E(x^{a+L}, x^{2L})$	$\varepsilon^{\infty}_a E(x^{a+L}, -x^L)$
ξ_a	$a(a+1)/2$	$a^2/4$	0	$a/2$
$D^{(1)}_{L+2}$ model[c]				
$p^{1/2}$	$-\exp(-\varepsilon/2L)$	$\exp(-\varepsilon/2L)$	$\exp(-\varepsilon/2L)$	$-\exp(-\varepsilon/2L)$
x	$\exp(-4\pi^2/\varepsilon)$	$\exp(-4\pi^2/\varepsilon)$	$\exp(-4\pi^2/\varepsilon)$	$\exp(-4\pi^2/\varepsilon)$
q	x^{L-2}	x^{L-2}	x^2	x^2
σ	$+1$	-1	$+1$	-1
u_a	$\varepsilon^L_a E(-x^{a+L/2}, x^L)$	$\varepsilon^L_a E(-x^a, x^L)$	$\varepsilon^L_a E(-x^a, x^L)$	$\varepsilon^L_a E(-x^{a+L/2}, x^L)$
ξ_a	$a(a-L+2)/4$	$a(a-L)/4$	0	$a/2$
$A^{(1)}_{L-1}$ model				
$p^{1/2}$	$-\exp(-\varepsilon/2L)$	$\exp(-\varepsilon/2L)$	$\exp(-\varepsilon/2L)$	$-\exp(-\varepsilon/2L)$
x	$\exp(-4\pi^2/\varepsilon)$	$\exp(-4\pi^2/\varepsilon)$	$\exp(-4\pi^2/\varepsilon)$	$\exp(-4\pi^2/\varepsilon)$
q	x^{L-2}	x^{L-2}	x^2	x^2
σ	$+1$	-1	$+1$	-1
u_a	$E(-x^{a+L/2}, x^L)$	$E(-x^a, x^L)$	$E(-x^a, x^L)$	$E(-x^{a+L/2}, x^L)$
ξ_a	$a(a-L+2)/4$	$a(a-L)/4$	0	$a/2$

[a] For the A_{L-1} model the parameter ε (>0) is defined in the first row through the relation with p. The quantity $x^{\xi_a} u_a$ is invariant under the change $a \to L-a$.

[b] For the D_{L+1} model $u_0 = u_{\bar{0}}$ and $\xi_0 = \xi_{\bar{0}}$.

[c] For the $D^{(1)}_{L+2}$ model $u_a = u_{\bar{a}}$, $\xi_a = \xi_{\bar{a}}$. The quantity $x^{\xi_a} u_a$ is invariant under the change $a \to a^*$ [see (1.30b)].

Table II. The Quantities G_a and F in (2.3) for Each Model and Regime[a]

Models	Regimes	G_a	F
A_{L-1}	I, IV	$w^{a(a-L)/2L}$	$x^{u(2u+2-L)/2L}$
	II, III	$w^{a(a-L)/4L}$	$x^{u(u+1)/2L}$
D_{L+1}	I, IV	$w^{a^2/4L}$	$x^{u(u+1-L)/2L}$
	II, III	$w^{a^2/8L}$	$x^{u(u+1)/4L}$
$D^{(1)}_{L+2}$, $A^{(1)}_{L-1}$	I, IV	$w^{a(a-L)/4L}$	$x^{u(u+1-L)/2L}$
	II, III	$w^{a(a-L)/4L}$	$x^{u(u+1)/2L}$

[a] In the D_{L+1} and $D^{(1)}_{L+2}$ models we assume that $G_a = G_{\bar{a}}$.

The diagonal $U(a, c)$ has the form by which the weight function $H(a, b, c)$ is determined as follows:

$$U(a, c)_{b,d} = \delta_{b,d}\, w^{\zeta H(a,b,c)} \tag{2.4}$$

where $\zeta = 1$ in regimes I and IV and $\zeta = -1$ in regimes II and III. The other matrices $U(a, c)$ listed in Table III may be diagonalized by following the argument in Section 3.3 of ref. 8. After relabeling the state variables suitably we have an equation of the form (2.4). This yields the full weight function $H(a, b, c)$ in (2.5)–(2.13c). Thus, if the face transfer matrix $U(a, c)$ has the nondiagonal limit, the symmetries of the Boltzmann weight $W(b, c, d, a)$ are lost from $H(a, b, c)$ through this diagonalization procedure.

Below we present the resulting forms for $H(a, b, c)$ assuming that the pairs (a, b) and (b, c) are admissible.

Table III. The Nondiagonal Matrices $U(a, c)$ Whose (b, d)-Element is Defined by (2.3)[a]

	Regimes I, IV	Regimes II, III
A_{L-1}	$U(L/2, L/2)$ (L even)	—
D_{L+1}	—	$U(1, 1)$
$D^{(1)}_{L+2}$	$U(1, 1)$, $U(L-1, L-1)$, $U(L/2, L/2)$ (L even)	—
$A^{(1)}_{L-1}$	$U(L/2), L/2)$ (L even)	$U(0, 0)$

[a] The $U(L/2, L/2)$ exists and is nondiagonal only for even L.

Regimes II and III

(a) A_{L-1} model:

$$H(a, b, c) = |a-c|/4 \tag{2.5}$$

(b) D_{L+1} model:

$$H(a, b, c) = |a-c|/4 \qquad \text{if} \quad a, b, c \neq \bar{0} \tag{2.6a}$$

$$H(1, \bar{0}, 1) = 1/2 \tag{2.6b}$$

$$H(a, b, c) = H(c, b, a) = H(\bar{a}, b, c) \tag{2.6c}$$

where $\bar{a}$ is defined in (1.28).

(c) $D^{(1)}_{L+2}$ model:

$$H(a, b, c) = |a-c|/4 \qquad \text{if} \quad a, b, c \neq \bar{0}, \bar{L} \tag{2.7a}$$

$$H(0, 1, \bar{0}) = 1 \tag{2.7b}$$

$$H(a, b, c) = H(c, b, a) = H(a^*, b^*, c^*) = H(\bar{a}, \bar{b}, \bar{c}) \tag{2.7c}$$

where a^* and $\bar{a}$ are defined in (1.30b) and (1.31b), respectively.

(d) $A^{(1)}_{L-1}$ model:

$$\begin{aligned} H(a, b, c) &= |a-c|/4 \\ &\text{if} \quad (a, b, c) \neq (1, 0, L-1), (L-1, 0, 1), (0, L-1, 0), \\ &\qquad (0, L-1, L-2), (L-2, L-1, 0) \end{aligned} \tag{2.8a}$$

$$H(1, 0, L-1) = 0, \quad H(0, L-1, 0) = 1, \quad H(0, L-1, L-2) = 1/2 \tag{2.8b}$$

$$H(a, b, c) = H(c, b, a) \tag{2.8c}$$

Regimes I and IV. In regimes I and IV, the function $H(a, b, c)$ depends on the integer part of $L/2$. We denote this by n:

$$n = [L/2] \tag{2.9}$$

(a) A_{L-1} model:

$$\begin{aligned} H(a, b, c) &= \min(n-b, (a-b+1)/2) \\ &\qquad \text{if} \quad b \leqslant n \text{ and } a \leqslant c \end{aligned} \tag{2.10a}$$

$$H(a, b, c) = H(c, b, a) = H(2n+1-a, 2n+1-b, 2n+1-c) \tag{2.10b}$$

(b) D_{L+1} model:

$$H(a\pm 1, a, a\mp 1)=0 \qquad \text{if} \quad a\neq\bar{0}, \quad 1\leqslant a\leqslant L-2 \tag{2.11a}$$

$$H(a, a+1, a)=1 \qquad \text{if} \quad a\neq\bar{0}, \quad 0\leqslant a\leqslant L-2 \tag{2.11b}$$

$$H(a, a-1, a)=0 \qquad \text{if} \quad a\neq\bar{0}, 1\leqslant a\leqslant L-1 \tag{2.11c}$$

$$H(0, 1, \bar{0})=0 \tag{2.11d}$$

$$H(a, b, c)=H(c, b, a)=H(\bar{a}, \bar{b}, \bar{c}) \tag{2.11e}$$

(c) $D^{(1)}_{L+2}$ model:

$$H(a, b, c)=\min(n-b, \tfrac{1}{2}(a-b+1))$$

$$\text{if} \quad a, b, c\neq\bar{0}, \bar{L}; b\leqslant n, a\leqslant c \tag{2.12a}$$

$$H(1, \bar{0}, 1)=H(L-1, \bar{L}, L-1)=1/2 \tag{2.12b}$$

$$\begin{aligned} H(a, b, c)&=H(c, b, a)=H(\bar{a}, b, c)\\ &=H(2n+1-a, 2n+1-b, 2n+1-c) \end{aligned} \tag{2.12c}$$

(d) $A^{(1)}_{L-1}$ model. The case $|a-b|=|b-c|=1$:

$$H(a, b, c)=\min(n-b, (a-b+1)/2)$$

$$\text{if} \quad b\leqslant n, \quad a\leqslant c \tag{2.13a}$$

$$H(a, b, c)=H(c, b, a)=H(2n+1-a, 2n+1-b, 2n+1-c) \tag{2.13b}$$

The case $|a-b|\neq 1$ or $|b-c|\neq 1$:

$$\begin{gathered} H(0, L-1, L-2)=H(0, L-1, 0)=0\\ H(L-1, 0, 1)=1/2, \qquad H(L-1, 0, L-1)=1\\ H(a, b, c)=H(c, b, a) \end{gathered} \tag{2.13c}$$

Except for the $A^{(1)}_{L-1}$ model with odd L, we have the following support property:

$$P_m(a\,|\,b, c)=0 \qquad \text{unless} \quad a-b\equiv m \bmod 2 \tag{2.14}$$

where $\bar{\lambda}\equiv\lambda \bmod 2$ is implied.

2.2. Background Configurations and Ground States

Now we specify the choice of (b, c) in taking the $m\to\infty$ limit of the quantity $P_m(a\,|\,b, c)$ in Section 2.1. For this purpose we introduce four sequences of the states $\Lambda^{(k)}=\{\Lambda^{(k)}_j\}_{j=1}^{\infty}$ $(1\leqslant k\leqslant 4)$ in the sequel.

The first one, $\Lambda^{(1)}$, is the alternating sequence which we shall consider for all the models A_{L-1}, D_{L+1}, $D^{(1)}_{L+2}$, and $A^{(1)}_{L-1}$:

$$\Lambda^{(1)}_{2j-1} = b, \qquad \Lambda^{(1)}_{2j} = c \tag{2.15}$$

where (b, c) is any admissible pair.

The second one, $\Lambda^{(2)}$, is relevant to the D_{L+1} and $D^{(1)}_{L+2}$ models and is defined by

$$\Lambda^{(2)}_{4j-3} = b, \qquad \Lambda^{(2)}_{4j-2} = c, \qquad \Lambda^{(2)}_{4j-1} = \bar{b}, \qquad \Lambda^{(2)}_{4j} = \bar{c} \tag{2.16}$$

where (b, c) is any admissible pair and $\bar{\lambda}$ in the model D_{L+1} (resp. $D^{(1)}_{L+2}$) is determined from λ by (1.28) [resp. (1.31b)]. Note that $\Lambda^{(2)}$ is identical with $\Lambda^{(1)}$ if $b = \bar{b}$, $c = \bar{c}$.

The third one, $\Lambda^{(3)}$, is relevant to regime II in the A_{L-1} model,

$$\Lambda^{(3)}_j = \langle b + j - 1 \rangle \tag{2.17}$$

where b is an arbitrary integer and the symbol $\langle x \rangle$ stands for the unique integer satisfying $1 \leqslant \langle x \rangle \leqslant L-1$, $\langle x \rangle - 1 \equiv \pm (x-1) \bmod 2(L-2)$.

The last one, $\Lambda^{(4)}$, is relevant to regime II in the D_{L+1} model,

$$\Lambda^{(4)}_j = \overline{\langle\!\langle b + j - 1 \rangle\!\rangle} \tag{2.18}$$

where b is an arbitrary integer and the symbol $\langle\!\langle x \rangle\!\rangle$ stands for the unique integer satisfying $0 \leqslant \langle\!\langle x \rangle\!\rangle \leqslant L-1$, $\langle\!\langle x \rangle\!\rangle \equiv \pm x \bmod 2(L-1)$. We specify $\overline{\langle\!\langle x \rangle\!\rangle}$ for $0 \leqslant \langle\!\langle x \rangle\!\rangle \leqslant L-1$ by (1.28). These sequences $\Lambda^{(k)}$ $(1 \leqslant k \leqslant 4)$ are admissible in the sense that all the pairs $(\Lambda^{(k)}_j, \Lambda^{(k)}_{j+1})$ therein are admissible.

The boundary states (b, c) in $P_m(a \mid b, c)$ are set to be the admissible pair $(\Lambda^{(k)}_{m+1}, \Lambda^{(k)}_{m+2})$ contained in the sequences $\Lambda^{(k)}$ $(1 \leqslant k \leqslant 4)$ described above. The choice of $\Lambda^{(k)}$ for each model and regime is specified in Table IV.

Table IV. The Background Configurations $\Lambda = \{\Lambda_j\}_{j=1}^{\infty}$ in Taking the $m \to \infty$ Limit of the Quantity $P_m(a \mid \Lambda_{m+1}, \Lambda_{m+2})$

	I	II	III	IV
A_{L-1}	$\Lambda^{(1)}$	$\Lambda^{(3)}$	$\Lambda^{(1)}$	$\Lambda^{(1)}$
D_{L+1}	$\Lambda^{(2)}$	$\Lambda^{(4)}$	$\Lambda^{(1)}$	$\Lambda^{(1)}$
$D^{(1)}_{L+2}$	$\Lambda^{(1)}$	$\Lambda^{(2)}$	$\Lambda^{(1)}$	$\Lambda^{(1)}$
$A^{(1)}_{L-1}$	$\Lambda^{(1)}$	$\Lambda^{(1)}$	$\Lambda^{(1)}$	$\Lambda^{(1)}$

For such choice of the background configuration Λ, the LSP is obtained by taking the $m \to \infty$ limit:

$$P(a \mid \Lambda) = \lim_{m \to \infty} P_m(a \mid \Lambda_{m+1}, \Lambda_{m+2}) \tag{2.19}$$

Among the background configurations, the ground states are defined to be the admissible sequences $(\lambda_1, \lambda_2, ..., \lambda_{m+1}, \lambda_{m+2})$ that minimize the "action" (2.2):

$$\sigma S_m(\lambda_1, ..., \lambda_{m+2}) \tag{2.20}$$

where σ is a sign factor given in Table I. We consider that the ground-state configuration in the original two-dimensional lattice is obtained by translating these one-dimensional sequences into the NE–SW direction except when the corner transfer matrices do not have the diagonal limit. See Section 2.1.

From (2.2d)–(2.13c) it is shown that the sequences $\Lambda^{(k)}$ $(1 \leqslant k \leqslant 4)$ in (2.15)–(2.18) become the ground states if the pair (b, c) satisfies the following conditions.

Regime I

(a) A_{L-1}, $D^{(1)}_{L+2}$ and $A^{(1)}_{L-1}$ models:

$$(b, c) = (n, n+1) \qquad \text{or} \qquad (n+1, n) \tag{2.21}$$

where n is defined in (2.9).

(b) D_{L+1} model:

$$(b, c) = (0, 1), (1, 0), (\bar{0}, 1), (1, \bar{0}) \tag{2.22}$$

Regime II. For arbitrary integer b, the sequences $\Lambda^{(3)}$, (2.17), and $\Lambda^{(4)}$, (2.18), become the ground states for A_{L-1} and D_{L+1} models, respectively. For $D^{(1)}_{L+2}$ and $A^{(1)}_{L-1}$ models the following are the conditions for $\Lambda^{(2)}$ and $\Lambda^{(1)}$, respectively, to become the ground states.

(a) $D^{(1)}_{L+2}$ model:

$$\begin{aligned}(b, c) = {} & (0, 1), (1, 0), (\bar{0}, 1), (1, \bar{0}), \\ & (L, L-1), (L-1, L), (\bar{L}, L-1), (L-1, \bar{L})\end{aligned} \tag{2.23}$$

(b) $A^{(1)}_{L-1}$ model:

$$(b, c) = (0, L-1), (L-1, 0) \tag{2.24}$$

Regime III. For arbitrary choice of the admissible pair (b, c), the sequence $\Lambda^{(1)}$, (2.15), is the ground state for A_{L-1} and $D^{(1)}_{L+2}$ models. For the other models the following conditions are imposed on (b, c).

(a) D_{L+1} model:

$$(b, c) \neq (1, \bar{0}), (\bar{0}, 1) \tag{2.25}$$

(b) $A^{(1)}_{L-1}$ model:

$$(b, c) \neq (0, L-1), (L-1, 0) \tag{2.26}$$

Regime IV. For arbitrary choice of the admissible pair (b, c), the sequence $\Lambda^{(1)}$, (2.15), is the ground state for the D_{L+1} model. For the other models the following conditions are imposed on (b, c).

(a) A_{L-1} and $A^{(1)}_{L-1}$ models:

$$(b, c) \neq (n, n+1), (n+1, n) \tag{2.27}$$

(b) $D^{(1)}_{L+1}$ model:

$$\begin{aligned}(b, c) \neq{} & (n, n+1), (n+1, n),\\ & (1, \bar{0}), (\bar{0}, 1), (L-1, \bar{L}), (\bar{L}, L-1)\end{aligned} \tag{2.28}$$

2.3. Results for LSPs

Below we summarize the results for LSPs. For all the models and regimes they take a product form:

$$P(a \mid \Lambda) = c_{a,\Lambda}(q)\, T_{a,\Lambda} \tag{2.29}$$

Here $c_{a,\Lambda}(q)$ symbolically denotes the $m \to \infty$ limit of the 1D configuration sum $X_m(a, b, c; q^{\pm 1})$ (up to some power corrections) and $T_{a,\Lambda}$ is the ratio of theta functions $\Theta^{(\pm, \pm)}_{j,k}(x, x^2)$ (see Appendix A.1). The line of the argument to obtain the LSPs goes as follows. In Section 3.2 we show that the $c_{a,\Lambda}(q)$ is given as the branching coefficient[7] (or linear combinations thereof) appearing in appropriate theta function identities. Suitable specializations of these identities are adjusted to (2.2b) in the limit $m \to \infty$ by using (A.9) to identify u_a in Table I. Dividing by the lhs, we get the LSP as a summand appearing in the rhs: $1 = \sum_{a \in S} P(a \mid \Lambda)$. In what follows the parameters x and $q = e^{2\pi i \tau}$ are specified in Table I.

Regime I. Relevant theta function identities for A_{L-1}, D_{L+1}, $D^{(1)}_{L+2}$ (L even) and $A^{(1)}_{L-1}$ (L even) models are (A.13) with the specialization

$z=\sqrt{x}$ and $(\varepsilon, l)=(-1, L)$, $(-1, 2L)$, $(1, L)$, and $(1, L)$, respectively. For $D^{(1)}_{L+2}$ and $A^{(1)}_{L-1}$ models with odd L, we also use (A.14).

(a) A_{L-1} model. The LSP becomes independent of the boundary states, showing that the system is disordered. Thus, we write $P(a)$ for $P(a \mid A^{(1)})$:

$$P(a)=c^{(-)}_{a,L}(q)\,T_a \tag{2.30a}$$

$$T_a=\frac{2\Theta^{(-,-)}_{a,L/2}(x,x^2)\,\eta(\tau)}{\Theta^{(-,-)}_{L/2-1,L/2-1}(x,x)^2\,\Theta^{(+,+)}_{\mu/2,1}(x,x^2)} \tag{2.30b}$$

where $c^{(-)}_{a,L}(q)$ is defined in (A.10) and $\mu=0$ or 1 according as L is even or odd.

(b) D_{L+1} model. The LSP $P(a \mid A^{(2)})$ is independent of the boundary states for (b, c) other than (2.22). Denoting it by $P(a)$, we have

$$P(a)=P(\bar{a})=c^{(-)}_{L+a,2L}(q)\,T_a \tag{2.31a}$$

$$T_a=\frac{2(\varepsilon^{\infty}_a)^2\,\Theta^{(-,-)}_{L+a,L}(x,x^2)\,\eta(\tau)}{\Theta^{(-,-)}_{L-1,L-1}(x,x^2)\,\Theta^{(+,+)}_{0,1}(x,x^2)} \tag{2.31b}$$

For (b,c) given in (2.22), we write $P(a \mid b, c)=P(a \mid A^{(2)})$ with $b=A^{(2)}_1$ and $c=A^{(2)}_2$:

$$\tfrac{1}{2}\,[P(a \mid b, c)+P(\bar{a} \mid b, c)]=c^{(-)}_{L+a,2L}(q)\,T_a \tag{2.31c}$$

$$\tfrac{1}{2}\,[P(0 \mid 0, 1)-P(\bar{0} \mid 0, 1)]=\frac{n(\tau)}{\eta(2\tau)}\,T_0 \tag{2.31d}$$

$$P(a \mid b, c)=P(\bar{a} \mid \bar{b}, \bar{c})=P(a \mid b, \bar{c}) \tag{2.31e}$$

where T_a is given by (2.31b).

(c) $D^{(1)}_{L+2}$ model. As in the A_{L-1} model, the LSP $P(a \mid A^{(1)})$ becomes independent of the boundary states (b, c). Denoting it simply by $P(a)$, we have

$$P(a)=P(\bar{a})=(\varepsilon^L_a)^2\,c^{(+)}_{a/2,L/2}(q)\,T_a \tag{2.32a}$$

$$T_a=\frac{\Theta^{(+,+)}_{a+L/2,L}(x,x^2)\,\eta(\tau)}{\Theta^{(+,+)}_{L-2,L-2}(x,x^2)\,\Theta^{(+,+)}_{L/2-\mu,2}(x,x^2)} \tag{2.32b}$$

where $c^{(+)}_{j,k}(q)$ is defined in (A.10) and $\mu=0$ or 1 according as a is even or odd, respectively.

(d) $A^{(1)}_{L-1}$ model. The case L is even: The LSP is independent of the boundary states. Denoting it by $P(a)$, we have

$$P(a)=c^{(+)}_{a/2,L/2}(q)\,T_a \tag{2.33a}$$

where T_a is given by (2.32b) with $\mu = 0$ or 1 according as a is even or odd, respectively.

The case L is odd: The boundary state dependence enters the LSP through the parity of b. Thus we write $P^{(\pm)}(a) = P(a \mid A^{(1)})$ corresponding to $a - b \equiv (1 \mp 1)/2 \bmod 2$. Then we have

$$P^{(\pm)}(a) = c^{(+)}_{L/2 \pm a, 2L}(q)\, T_{\pm a} \tag{2.33b}$$

where T_a is given by (2.32b) with $\mu = 0$ or 1 according as $a + (1 \mp 1)/2$ is even or odd, respectively.

Regime II. In the A_{L-1} and D_{L+1} models the LSPs are expressed in terms of the Hecke modular functions. Necessary theta function identities involving them can be found in Appendix B.4 in ref. 7. The LSPs in $D^{(1)}_{L+2}$ and $A^{(1)}_{L-1}$ are similar to regime I results and are obtained by using (A.15).

(a) A_{L-1} model. The boundary state dependence of the LSP $P(a \mid A^{(3)})$ is specified by b in (2.17). Thus we simply denote it by $P(a \mid b)$:

$$P(a \mid b) = e^{L-2}_{b-1, a-1}(\tau)\, T_a \tag{2.34a}$$

$$T_a = x^{L/8}\, \Theta^{(-,+)}_{a,L}(x, x^2)/\eta(\tau) \tag{2.34b}$$

where $e^{l}_{j,k}(\tau)$ is the Hecke modular function described in Appendix B in ref. 7.

(b) D_{L+1} model. Much the same as for the A_{L-1} model, the boundary state dependence enters the LSP $P(a \mid A^{(4)})$ only through the integer b in (2.18). Thus we denote it by $P(a \mid b)$:

$$P(a \mid b) = P(\bar{a} \mid b) = \varepsilon^{\infty}_a \left[e^{2L-2}_{L+b-1, L+a-1}(\tau) + e^{2L-2}_{L+b-1, L-a-1}(\tau) \right] T_a \tag{2.35a}$$

$$T_a = \varepsilon^{\infty}_a\, x^{L/4}\, \Theta^{(-,+)}_{L+a, 2L}(x, x^2)/\eta(\tau) \tag{2.35b}$$

(c) $D^{(1)}_{L+2}$ model. For (b, c) other than (2.23), the LSP $P(a \mid A^{(2)})$ is independent of the boundary states. Denoting it by $P(a)$, we have

$$P(a) = P(\bar{a}) = \varepsilon^{L}_a\, c^{(+)}_{L+a, 2L}(q)\, T_a \tag{2.36a}$$

$$T_a = \frac{2\varepsilon^{L}_a\, \Theta^{(+,+)}_{a,L}(x, x^2)\, \eta(\tau)}{\Theta^{(+,+)}_{L-2, L-2}(x, x^2)\, \Theta^{(+,+)}_{\mu,2}(x, x^2)} \tag{2.36b}$$

Here $\mu = 0$ or 1 according as a (or $\bar{a}$) is even or odd and $c^{(+)}_{j,k}(q)$ is defined in (A.10).

For (b, c) given in (2.23), we write $P(a \mid b, c)=P(a \mid \Lambda^{(2)})$ with $b=\Lambda_1^{(2)}$ and $c=\Lambda_2^{(2)}$:

$$\tfrac{1}{2}\left[P(a \mid b, c)+P(\bar{a} \mid b, c)\right]=\varepsilon_a^L c_{L+a, 2L}^{(+)}(q) T_a \tag{2.36c}$$

$$P(0 \mid 0, 1)-P(\bar{0} \mid 0, 1)=\frac{\eta(\tau)}{\eta(2\tau)} T_0 \tag{2.36d}$$

$$\begin{aligned} P(a \mid b, c) &= P(\bar{a} \mid \bar{b}, \bar{c})=P(a^* \mid b^*, c^*) \\ &= P(\bar{a} \mid b, c) \qquad \text{if} \quad a, \bar{a} \neq b, c \end{aligned} \tag{2.36e}$$

Here T_a is given by (2.36b) with $\mu=0$ or 1 according as a is even or odd, and $\bar{\lambda}$ and λ^* are specified in (1.30b) and (1.31b).

(d) $A_{L-1}^{(1)}$ model. The case L is even: The LSP is independent of the boundary states, showing that the system is disordered. Thus we write $P(a)$ for $P(a \mid \Lambda^{(1)})$:

$$\begin{aligned} 2\varepsilon_a^L P(a) &= \left[c_{L-a, 2L}^{(+)}(q)+c_{a, 2L}^{(+)}(q)\right] T_a \\ &= c_{L/4-a/2, L/2}^{(+)}(q) T_a \end{aligned} \tag{2.37a}$$

where T_a is given by (2.36b) with $\mu=0$ or 1 according as a is even or odd.

The case L is odd: Let $\hat{b}$ be a unique integer satisfying $1 \leqslant \hat{b} \leqslant L$, $\hat{b} \equiv b \bmod L$. The boundary state dependence enters the LSP through the parity of $\hat{b}$. Thus we write $P^{(\pm)}(a)=P(a \mid \Lambda^{(1)})$ corresponding to $a-\hat{b} \equiv (1 \mp 1)/2 \bmod 2$. Then we have

$$2\varepsilon_a^L P^{(\pm)}(a)=c_{L/2 \pm(a-L/2), 2L}^{(+)}(q) T_a \tag{2.37b}$$

where T_a is given by (2.36b) with $\mu=0$ or 1 according as $a+(1 \pm 1)/2$ is even or odd, respectively.

In the remaining regimes III and IV, the background configuration is set to be $\Lambda^{(1)}$, (2.15), for all the models. See Table IV. Thus we write the LSP $P(a \mid b, c)=P(a \mid \Lambda^{(1)})$ with $b=\Lambda_1^{(1)}$ and $c=\Lambda_2^{(1)}$. We shall use the parameters r and s defined by

$$r=\frac{b+c-1}{2}, \qquad s=\frac{b-c+1}{2}+1 \tag{2.38}$$

When $b=c \pm 1$, these are integers: $r=b$ or $b-1$, $s=1$ or 2.

Regime III. For all the models the relevant theta function identities are provided by (A.16) ($\varepsilon_2=+1$) with the specialization $z=x$, $q=x^2$.

(a) A_{L-1} model:

$$P(a\,|\,b,c)=c^{(-,+)}_{r,s,a}(x^2)\,T_{r,s,a} \tag{2.39a}$$

$$T_{r,s,a}=\frac{\Theta^{(-,+)}_{1,2}(x,x^2)\,\Theta^{(-,+)}_{a,L}(x,x^2)}{\Theta^{(-,+)}_{r,L-1}(x,x^2)\,\Theta^{(-,+)}_{s,3}(x,x^2)} \tag{2.39b}$$

Here r, s are given by (2.38) and $c^{(-,+)}_{j_1j_2j_3}(q)$ is defined by (A.16) with m_1, m_2, m_3 specified as follows:

$$m_1=L-1,\qquad m_2=3,\qquad m_3=L \tag{2.40}$$

(b) D_{L+1} model: The LSP enjoys the symmetry

$$P(a\,|\,b,c)=P(\bar a\,|\,b,c)=P(a\,|\,\bar b,c)=P(a\,|\,b,\bar c) \tag{2.41}$$

Thus we assume without loss of generality that a, b, $c\neq\bar 0$:

$$P(a\,|\,b,c)=\varepsilon_a^\infty[c^{(-,+)}_{L+r,s,L-a}(x^2)+c^{(-,+)}_{L+r,s,L+a}(x^2)]\,T_{r,s,a} \tag{2.42a}$$

$$T_{r,s,a}=\varepsilon_a^\infty\,\frac{\Theta^{(-,+)}_{1,2}(x,x^2)\,\Theta^{(-,+)}_{L+a,2L}(x,x^2)}{\Theta^{(-,+)}_{L+r,2L-1}(x,x^2)\,\Theta^{(-,+)}_{s,3}(x,x^2)} \tag{2.42b}$$

Here r, s are given in (2.38) and $c^{(-,+)}_{j_1j_2j_3}(q)$ is defined by (A.16) with m_1, m_2, m_3 taking the following values:

$$m_1=2L-1,\qquad m_2=3,\qquad m_3=2L \tag{2.43}$$

(c) $D^{(1)}_{L+2}$ model. As in regime II, the LSP has the symmetry (2.36e) and the results are reduced to the following:

$$\tfrac{1}{2}[P(a\,|\,b,c)+P(\bar a\,|\,b,c)]=c^{(+,+)}_{r,s,a}(x^2)\,T_{r,s,a},\qquad \text{for}\quad a,b,c\neq\bar 0,\bar L \tag{2.44a}$$

$$\tfrac{1}{2}[P(0\,|\,0,1)-P(\bar 0\,|\,0,1)]=\frac{\eta(\tau)}{\eta(2\tau)}\,T_{0,1,0} \tag{2.44b}$$

$$T_{r,s,a}=\varepsilon_a^L\,\frac{\Theta^{(-,+)}_{1,2}(x,x^2)\,\Theta^{(+,+)}_{a,L}(x,x^2)}{\Theta^{(+,+)}_{r,L-1}(x,x^2)\,\Theta^{(-,+)}_{s,3}(x,x^2)} \tag{2.44c}$$

In (2.44) r and s are given by (2.38) and $c^{(+,+)}_{j_1j_2j_3}(q)$ is defined in (A.16) with m_1, m_2, m_3 specified by (2.40).

(d) $A^{(1)}_{L-1}$ model:

$$(\varepsilon_a^L)^2\,P(a\,|\,b,c)=\tfrac{1}{2}\,[c^{(+,+)}_{r,s,a}(x^2)+c^{(+,+)}_{r,s,L-a}(x^2)$$
$$+\varepsilon_a^L[c^{(-,+)}_{r,s,a}(x^2)-c^{(-,+)}_{r,s,L-a}(x^2)]\,T_{r,s,a}$$
$$\text{if}\quad (b,c)\neq(0,L-1),\,(L-1,0) \tag{2.45a}$$

$$P(a\,|\,0,L-1)=P(a\,|\,0,1),\quad P(a\,|\,L-1,0)=P(a\,|\,L-1,L-2) \tag{2.45b}$$

Here r and s are given in (2.38) and $T_{r,s,a}$ in (2.44c). The functions $c^{(\pm,+)}_{j_1 j_2 j_3}(q)$ are defined by (A.16) with m_1, m_2, m_3 specified by (2.40). Note that in the case L is odd, either $c^{(\pm,+)}_{r,s,a}(q)$ or $c^{(\pm,+)}_{r,s,L-a}(q)$ is zero because of (A.18b).

Regime IV. The LSPs in this regime are also related to the identity (A.16) with the specialization $z=x$, $q=x^2$.

(a) A_{L-1} model. Define the integers n by (2.9) and r, s by (2.38). We assume that $b+c\leqslant 2n-1$. The other cases are reduced to this case as follows:

$$\begin{aligned} P(a\,|\,b,c) &= P(a\,|\,n,n-1) && \text{if}\quad (b,c)=(n,n+1)\\ &= P(L-a\,|\,L-n-1,L-n-2) && \text{if}\quad (b,c)=(n,n+1)\\ &= P(L-a\,|\,L-b,L-c) && \text{if}\quad b+c\geqslant 2n+3 \end{aligned}\tag{2.46}$$

The case L is even:

$$\varepsilon_a^{L/2}[P(a\,|\,b,c)\pm P(L-a\,|\,b,c)] = c^{(-,\mp)}_{r,s,a}(x^2)\,T_{r,s,a}\tag{2.47a}$$

$$T_{r,s,a} = \frac{\Theta^{(-,+)}_{1,2}(x,x^2)\,\Theta^{(-,-)}_{a,L/2}(x,x^2)}{\Theta^{(-,-)}_{r,L/2-1}(x,x^2)\,\Theta^{(-,+)}_{s,3}(x,x^2)}\tag{2.47b}$$

The case L is odd:

$$P(a\,|\,b,c) = c^{(-,-)}_{r,s,a}(x^2)\,T_{r,s,a}\tag{2.47c}$$

where $T_{r,s,a}$ is given by (2.47b). The functions $c^{(-,\pm)}_{r,s,a}(q)$ in (2.47a) and (2.47c) are defined by (A.16) with m_1, m_2, m_3 taking the following values:

$$m_1 = L/2-1,\qquad m_2=3,\qquad m_3=L/2\tag{2.48}$$

(b) D_{L+1} model. The LSP enjoys the symmetry

$$P(a\,|\,b,c) = P(\bar{a}\,|\,\bar{b},\bar{c})\tag{2.49}$$

Thus the results are reduced to the following:

$$P(a\,|\,b,c) = \varepsilon_a^{\infty} c^{(-,-)}_{L-1-r,3-s,L-a}(x^2)\,T_{r,s,a}\qquad \text{for}\quad a,b,c\neq\bar{0}\tag{2.50a}$$

$$\tfrac{1}{2}\,[P(0\,|\,0,1)-P(\bar{0}\,|\,0,1)] = \frac{\eta(\tau)}{\eta(2\tau)}\,T_{0,1,0}\tag{2.50b}$$

$$T_{r,s,a} = \varepsilon_a^{\infty}\,\frac{\Theta^{(-,+)}_{1,2}(x,x^2)\,\Theta^{(-,-)}_{L-a,L}(x,x^2)}{\Theta^{(-,-)}_{L-1-r,L-1}(x,x^2)\,\Theta^{(-,+)}_{3-s,3}(x,x^2)}\tag{2.50c}$$

Here r and s are given in (2.38) and the $c^{(\pm,-)}_{j_1j_2j_3}(q)$ are defined by (A.16) with m_1, m_2, m_3 specified by (2.40).

(c) $D^{(1)}_{L+2}$ model. The LSP has the symmetry (2.41). Thus we assume with no loss of generality $a, b, c \neq \bar{0}, \bar{L}$. Below we give the results assuming that $b+c \leqslant 2n-1$. The other cases are reduced to this case by (2.46).

The case L is even:

$$\begin{aligned}\varepsilon^L_{a+L/2} & P(a \mid b, c) \\ &= \tfrac{1}{2}\,\varepsilon^L_a [c^{(+,+)}_{r+L/2,s,a+L/2}(x^2) + c^{(+,+)}_{r+L/2,s,a-L/2}(x^2) \\ &\quad + c^{(-,+)}_{r+L/2,s,a+L/2}(x^2) - c^{(-,+)}_{r+L/2,s,a-L/2}(x^2)]\, T_{r+L/2,s,a+L/2}\end{aligned} \tag{2.51a}$$

The case L is odd:

$$\begin{aligned}P(a \mid b, c) = \varepsilon^L_a [& c^{(+,+)}_{r+L/2,s,a+L/2}(x^2) \\ & + c^{(+,+)}_{r+L/2,s,a-L/2}(x^2)]\, T_{r+L/2,s,a+L/2}\end{aligned} \tag{2.51b}$$

In (2.51) r and s are given by (2.38) and $T_{r+L/2,s,a+L/2}$ is obtained from (2.44c). The function $c^{(\pm,+)}_{j_1j_2j_3}(q)$ is defined by (A.16) with m_1, m_2, m_3 taking the values (2.40).

(d) $A^{(1)}_{L-1}$ model. The LSP $P(a \mid b, c)$ has the property (2.46) if $(b, c) \neq (0, L-1)$, $(L-1, 0)$. The cases $(b, c) = (0, L-1)$ or $(L-1, 0)$ are reduced to the above cases as follows.

The case L is even

$$\begin{aligned} & P(a \mid L-1, 0) = P(L-a \mid 1, 0) && \text{for} \quad a \in 2Z+1, \quad 1 \leqslant a \leqslant L-1 \\ & P(a \mid 0, L-1) = P(L-a \mid 0, 1) && \text{for} \quad a \in 2Z, \quad 2 \leqslant a \leqslant L-1 \\ & P(0 \mid 0, L-1) = P(0 \mid 0, 1) \end{aligned} \tag{2.52a}$$

The case L is odd:

$$\begin{aligned} & P(a \mid L-1, 0) = P(L-a \mid 1, 0) && \text{for} \quad 1 \leqslant a \leqslant L-1 \\ & P(0 \mid L-1, 0) = P(0 \mid 1, 0) \\ & P(a \mid 0, L-1) = P(L-a \mid 0, 1) && \text{for} \quad 1 \leqslant a \leqslant L-1 \\ & P(0 \mid 0, L-1) = P(0 \mid 0, 1) \end{aligned} \tag{2.52b}$$

In view of this, we give the results assuming that $(b, c) \neq (0, L-1)$, $(L-1, 0)$ and $b+c \leqslant 2n-1$.

The case L is even:

$$\begin{aligned}
&\varepsilon^L_{a+L/2}\varepsilon^L_a P(a\,|\,b,c)\\
&\quad=\tfrac{1}{2}\,[c^{(+,+)}_{r+L/2,s,a+L/2}(x^2)+c^{(+,+)}_{r+L/2,s,a-L/2}(x^2)\\
&\qquad c^{(-,+)}_{r+L/2,s,a+L/2}(x^2)+c^{(-,+)}_{r+L/2,s,a-L/2}(x^2)]\,T_{r+L/2,s,a+L/2}
\end{aligned}\tag{2.53a}$$

The case L is odd:

$$\begin{aligned}
\varepsilon^L_a P(a\,|\,b,c)&=c^{(+,+)}_{r+L/2,s,a\pm L/2}(x^2)\,T_{r+L/2,s,a+L/2}\\
&\qquad\text{if}\quad a-b\equiv(1\mp1)/2 \bmod 2
\end{aligned}\tag{2.53b}$$

Here r and s are defined in (2.38) and $T_{r+L/2,s,a+L/2}$ is given by using (2.44c). The function $c^{(\pm,+)}_{j_1 j_2 j_3}(q)$ is defined by (A.16) with m_1, m_2, m_3 taking the values (2.40).

3. ONE-DIMENSIONAL CONFIGURATION SUMS

This section is devoted to the study of the one-dimensional configuration sums $X_m(a,b,c;q)$ introduced in Section 2.1. They are the quantities of primary importance in the analysis of the LSPs and become the generating functions for the eigenvalue spectrum of corner transfer matrices in the limit of lattice size m large. For finite m they are q-polynomials, while in the limit $m\to\infty$ they tend to modular functions (up to a power of q). One may regard m as discrete time and consider the $X_m(a,b,c;q)$ of (2.2c) and (2.2d) as a sort of functional integral for a particle moving about the diagrams in Figs. 1–4.

3.1. Expressions in Terms of Gaussian Polynomials

Here we rewrite the 1D configuration sum $X_m(a,b,c;q)$ of (2.2c) and (2.2d) in a form that is suitable for taking the limit $m\to\infty$. The results amount to a series involving Gaussian polynomials[21]:

$$\begin{aligned}
\begin{bmatrix} M\\ N\end{bmatrix}&=\prod_{j=1}^{N}\frac{1-q^{M-N+j}}{1-q^j}\qquad\text{for}\quad 0\leqslant N\leqslant M\\
&=0\qquad\text{otherwise}
\end{aligned}\tag{3.1}$$

Our strategy is to use the linear difference equation and the initial condition that completely characterize the $X_m(a,b,c;q)$ of (2.2c) and (2.2d):

$$X_m(a,b,c;q)=\sum_d{}'\,X_{m-1}(a,d,b;q)\,q^{mH(d,b,c)}\tag{3.2a}$$

$$X_0(a,b,c;q)=\delta_{a,b}\tag{3.2b}$$

Here the sum in (3.2a) is taken over d such that the pair (d, b) is admissible. For each model there are two cases to consider, depending on the forms of the weight function $H(a, b, c)$ in (2.5)–(2.13). We denote them by $X_m(a, b, c; q^{\pm 1})$ in regimes II and III and $Y_m(a, b, c; q^{\pm 1})$ in regimes I and IV.

Regimes II and III. We exploit the method developed in refs. 7 and 8, and construct the solution to (3.2) from the function $f_m(b, c; q)$ $(b, c \in Z)$ that satisfies the simplified equation of the form

$$f_m(b, c; q) = \sum_{d = b \pm 1} f_{m-1}(d, b; q)\, q^{m|d-c|/4} \tag{3.3a}$$

$$f_0(b, c; q) = \delta_{b,0} \tag{3.3b}$$

We set $f_m(b, c; q) = 0$ unless $|b - c| = 1$. By the definition, $f_m(b, c; q)$ has the following properties:

$$f_m(b, c; q) = 0 \qquad \text{unless} \quad b \equiv m \bmod 2, \quad |b| \leqslant m \tag{3.4a}$$

$$f_m(b, c; q) = f_m(-b, -c; q) \tag{3.4b}$$

An explicit formula is available in terms of Gaussian polynomials:

$$f_m(b, c; q) = q^{bc/4} \begin{bmatrix} m \\ (m+b)/2 \end{bmatrix} \tag{3.5}$$

The solution to Eq. (3.2) is built basically by making the following linear superposition of the function $f_m(b, c; q)$:

$$F_m^{(L)}(a, b, c; q) = \sum_{v \in Z} q^{-Lv^2 + (L/2 - a)v + a/4} \times f_m(b - a - 2Lv, c - a - 2Lv; q) \tag{3.6}$$

From (3.4a) we see that $F_m^{(L)}(a, b, c; q)$ [or $q^{1/2} F_m^{(L)}(a, b, c; q)$] is a polynomial in q having the property

$$F_m^{(L)}(a, b, c; q) = 0 \qquad \text{unless} \quad a - b \equiv m \bmod 2 \tag{3.7}$$

Below we list the results for each model. For simplicity, we shall suppress the argument q.

(a) A_{L-1} model

$$X_m(a, b, c) = q^{-a/4}[F_m^{(L)}(a, b, c) - F_m^{(L)}(-a, b, c)] \tag{3.8}$$

Much the same as the Boltzmann weights (1.25), the 1D configuration sum enjoys the symmetry

$$X_m(a, b, c) = X_m(L-a, L-b, L-c) \tag{3.9}$$

(b) D_{L+1} model. The 1D configuration sum is neatly expressed in terms of that for the A_{2L-1} model. Let us denote the polynomial (3.8a) by $\tilde{X}_m^{(L)}(a, b, c)$. Then we have

$$X_m(a, b, c) = \varepsilon_a^\infty [\tilde{X}_m^{(2L)}(L+a, L+b, L+c) + \tilde{X}_m^{(2L)}(L-a, L+b, L+c)]$$
$$\text{if} \quad a \neq \bar{0}, \quad (b, c) \neq (0, 1), (\bar{0}, 1), (1, \bar{0}) \tag{3.10}$$

where the symbol ε_a^∞ is defined in (1.8). The remaining cases are reduced to (3.10) as follows:

$$X_m(\bar{0}, b, c) = X_m(0, b, c) \qquad \text{for all} \quad (b, c) \tag{3.11a}$$

$$X_m(a, 0, 1) = X_{m-1}(a, 1, 0) \tag{3.11b}$$

$$X_m(a, \bar{0}, 1) = q^{m/2} X_{m-1}(a, 1, 0) \tag{3.11c}$$

$$X_m(a, 1, \bar{0}) = X_m(a, 1, 0) \tag{3.11d}$$

(c) $D_{L+2}^{(1)}$ model. The case $a, b, c \neq \bar{0}, \bar{L}, (b, c) \neq (1, 0), (L-1, L)$: Define an integer r by (2.38). Then we have

$$\tfrac{1}{2}[X_m(a, b, c) + X_m(\bar{a}, b, c)]$$
$$= \varepsilon_a^L \varepsilon_r^{L-1} q^{-a/4} [F_m^{(L)}(a, b, c) + F_m^{(L)}(-a, b, c)] \tag{3.12a}$$

$$X_m(0, 0, 1) - X_m(\bar{0}, 0, 1) = \prod_{j=1}^{m/2} (1 - q^{2j-1}) \qquad \text{if} \quad m \text{ is even} \tag{3.12b}$$

The other cases are reduced to this case as follows:

$$X_m(a, 1, 0) = X_{m+1}(a, 0, 1) \tag{3.13a}$$

$$X_m(a, b, c) = X_m(a^*, b^*, c^*) = X_m(\bar{a}, \bar{b}, \bar{c})$$
$$= X_m(\bar{a}, b, c) \qquad \text{if} \quad a, \bar{a} \neq b, c \tag{3.13b}$$

where a^* and $\bar{a}$ are defined in (1.30b) and (1.31b), respectively.

(d) $A_{L-1}^{(1)}$ model. The case $(b, c) \neq (L-1, 0), (0, L-1)$:

$$X_m(a, b, c) = q^{-a/4} F_m^{(L)}(a, b, c) + q^{-(L-a)/4} F_m^{(L)}(a-L, b, c) \tag{3.14}$$

The remaining cases are reduced to this by the relations

$$X_m(a, L-1, 0) = q^{m/2} X_m(a, L-1, L-2) \tag{3.15a}$$

$$X_m(a, 0, L-1) = X_m(a, 0, 1) \tag{3.15b}$$

We note that in the case L is odd, we have $F_m^{(L)}(a, b, c) = 0$ or $F_m^{(L)}(a-L, b, c) = 0$ because of (3.7).

Regimes I and IV. We follow a similar procedure as with regard to regimes II and III and consider the function $g_m(n; b, c; q)$ $(n, b, c \in Z)$ defined by the linear difference equation and the initial condition

$$g_m(n; b, c; q) = \sum_{d=b\pm 1} g_{m-1}(n; d, b; q)\, q^{mH(n, d, b, c)} \tag{3.16a}$$

$$g_0(n; b, c; q) = \delta_{b,0} \tag{3.16b}$$

Here the function $H(n, a, b, c)$ is the one given in (2.9)–(2.13c) and we have explicitly exhibited the n dependence. The following explicit formula is valid for a general integer n (not restricted to $n = [L/2]$ of (2.9)]:

$$\begin{aligned} g_m(n; b, c; q) &= \bar{f}_m(b, c) && \text{if} \quad b + c \leqslant 2n - 1 \\ &= \bar{f}_m(b, 2n-b-1) && \text{if} \quad (b, c) = (n, n+1) \\ &= \bar{f}_m(-n-1, -n-2) && \text{if} \quad (b, c) = (n+1, n) \\ &= \bar{f}_m(-b, -c) && \text{if} \quad b + c \geqslant 2n+3 \end{aligned} \tag{3.17a}$$

where the function $\bar{f}_m(b, c)$ is given by

$$\begin{aligned} \bar{f}_m(b, c) &= f_m(b, c; q^{-1})\, q^{m^2/4 - m(b-c-1)/4 - b/4} \\ &= q^{(m-b)(c-b+1)/4} \begin{bmatrix} m \\ (m+b)/2 \end{bmatrix} \end{aligned} \tag{3.17b}$$

By definition, $g_m(n; b, c; q)$ also has the support property

$$g_m(n; b, c; q) = 0 \qquad \text{unless} \quad b \equiv m \bmod 2, \quad |b| \leqslant m \tag{3.18}$$

We introduce two types of linear superposition of $g_m(n; b, c; q)$ as follows:

$$\begin{aligned} G_m^{(L)}(a, b, c; q) = \sum_{v \in Z} & q^{2Lv^2 + (2a-L)v - a/2} \\ & \times g_m(n-a-2Lv; b-a-2Lv, c-a-2Lv; q) \end{aligned} \tag{3.19a}$$

$$\begin{aligned} \tilde{G}_m^{(L)}(a, b, c; q) = \sum_{v \in Z} & q^{Lv^2 + (a-L/2)v - a/4} \\ & \times g_m(n-a-2Lv; b-a-2Lv, c-a-2Lv; q) \end{aligned} \tag{3.19b}$$

Up to an overall power $q^{1/2}$ these are polynomials in q having the property

$$G_m^{(L)}(a, b, c; q) = \tilde{G}_m^{(L)}(a, b, c; q) = 0 \qquad \text{unless} \quad a - b \equiv m \bmod 2 \tag{3.20}$$

The 1D configuration sums for A_{L-1} and D_{L+1} models are expressed in terms of $G_m^{(L)}(a, b, c; q)$, while those for $D_{L+2}^{(1)}$ and $A_{L-1}^{(1)}$ models are written as $\tilde{G}_m^{(L)}(a, b, c; q)$. Below we give the results assuming that $n = [L/2]$.

(a) A_{L-1} model:

$$Y_m(a, b, c) = q^{a/2}[G_m^{(L)}(a, b, c) - G_m^{(L)}(-a, b, c)] \tag{3.21}$$

From (3.17) and (3.21) we deduce the following properties:

$$\begin{aligned} Y_m(a, b, c) &= Y_m(a, n, n-1) && \text{if} \quad (b, c) = (n, n+1) \\ &= Y_m(L-a, L-n-1, L-n-2) && \text{if} \quad (b, c) = (n+1, n) \\ &= Y_m(L-a, L-b, L-c) && \text{if} \quad b + c \geqslant 2n+3 \end{aligned} \tag{3.22}$$

(b) D_{L+1} model. As in regimes II and III, the 1D configuration sum is expressed in terms of that for the A_{2L-1} model. Let us denote by $\tilde{Y}_m^{(L)}(a, b, c)$ the polynomial given in (3.21). Then we have

$$\begin{aligned} &\tfrac{1}{2}[Y_m(a, b, c) + Y_m(\bar{a}, b, c)] \\ &\quad = \varepsilon_a^{\infty}[\tilde{Y}_m^{(2L)}(L+a, L+b, L+c) + \tilde{Y}_m^{(2L)}(L+a, L-b, L-c)] \\ &\quad \text{if} \quad a, b, c \neq \bar{0}, \quad (b, c) \neq (0, 1) \end{aligned} \tag{3.23a}$$

$$Y_m(0, 0, 1) - Y_m(\bar{0}, 0, 1) = \prod_{j=1}^{m/2} (q^{2j-1} - 1) \qquad \text{if} \quad m \text{ is even} \tag{3.23b}$$

The other cases are reduced to (3.23) by the following relations:

$$Y_m(a, 0, 1) = Y_{m-1}(a, 1, 0) \tag{3.24a}$$

$$Y_m(a, b, c) = Y_m(\bar{a}, \bar{b}, \bar{c}) \tag{3.24b}$$

(c) $D_{L+2}^{(1)}$ model. The 1D configuration sum has the following properties:

$$Y_m(a, b, c) = Y_m(a, b, \bar{c}) = Y_m(\bar{a}, b, c) \tag{3.25a}$$

$$Y_m(a, \bar{0}, 1) = q^{-m/2} Y_m(a, 0, 1) = q^{m/2} Y_{m-1}(a, 1, 0) \tag{3.25b}$$

$$Y_m(a, \bar{L}, L-1) = q^{-m/2} Y_m(a, L, L-1) = q^{m/2} Y_{m-1}(a, L-1, L) \tag{3.25c}$$

In view of this we assume that $a, b, c \neq \bar{0}, \bar{L}$ and $(b, c) \neq (0, 1), (L, L-1)$. Then we have

$$Y_m(a, b, c) = \varepsilon_a^L q^{a/4}[\tilde{G}_m^{(L)}(a, b, c) + \tilde{G}_m^{(L)}(-a, b, c)] \tag{3.26}$$

We remark that the $Y_m(a, b, c)$ in (3.26) also has the properties (3.22).

(d) $A_{L-1}^{(1)}$ model. For $(b, c) \neq (0, L-1), (L-1, 0)$, we have

$$Y_m(a, b, c) = q^{a/4}\tilde{G}_m^{(L)}(a, b, c) + q^{(L-a)/4}\tilde{G}_m^{(L)}(a-L, b, c) \tag{3.27}$$

The cases $(b, c) = (L-1, 0)$ and $(0, L-1)$ are reduced to this as follows.

$$Y_m(a, L-1, 0) = Y_m(L-a, 1, 0) \tag{3.28a}$$

$$Y_m(a, 0, L-1) = Y_m(L-a, 0, 1) \tag{3.28b}$$

Note that the $Y_m(a, b, c)$ in (3.27) also enjoys the properties (3.22). We remark that in the case L is odd, the 1D configuration sum $Y_m(a, b, c)$ is nonzero for both cases $a-b \equiv m$ and $a-b \not\equiv m \bmod 2$.

3.2. 1D Configuration Sums As Modular Functions

We now proceed to the evaluation of $X_m(a, b, c; q^{\pm 1})$ and $Y_m(a, b, c; q^{\pm 1})$ in the limit of m large. Using the Gaussian polynomial representations in Section 3.1, it is straightforward to take the limit $m \to \infty$ (except in regime II for A_{L-1} and D_{L+1} models). The results turn out to be modular functions. They are described in the Appendix as the branching coefficients appearing in appropriate theta function identities. Here we present the results for the boundary states (b, c) satisfying the ground-state conditions (2.21)–(2.28). The computation for the other choice of (b, c) is no more difficult than for the ground-state case and the results take quite similar forms. We fix the parity of m to be even when the background configuration is $A^{(1)}$ [see (2.15)]. The odd m limit can be reduced to this case. Except for the $A_{L-1}^{(1)}$ model with odd L, we assume the support property

$$\begin{aligned} X_m(a, b, c, q^{\pm 1}) &= Y_m(a, b, c; q^{\pm 1}) \\ &= 0 \qquad \text{unless} \quad a-b \equiv m \bmod 2 \end{aligned} \tag{3.29}$$

Regime I. Relevant modular functions for A_{L-1} and D_{L+1} (resp. $D_{L+2}^{(1)}$ and $A_{L-1}^{(1)}$) models are $c_{j,k}^{(-)}(q)$ [resp. $c_{j,k}^{(+)}(q)$] in (A.10). In view of (2.21) we assume that $(b, c) = (n, n+1)$ or $(n+1, n)$ in A_{L-1}, $D_{L+2}^{(1)}$, and $A_{L-1}^{(1)}$ models.

(a) A_{L-1} model:

$$\lim_{m\ \text{even} \to \infty} Y_m(a, b, c; q) = \frac{E(q^a, q^L)}{\phi(q)} \tag{3.30}$$

where the functions $E(z, q)$ and $\phi(q)$ are defined in (1.3) and (1.6), respectively.

(b) D_{L+1} model. From (2.16) and (2.22) we see that the ground state is of period 4. Due to the symmetry (3.24b) it is enough to treat the cases $(b, c) = (0, 1)$ or $(1, 0)$. In both cases we have

$$\lim_{m \in 4Z \to \infty} Y_m(a, b, c; q) = E(q^{L+a}, q^{2L})/\phi(q) \qquad \text{if} \quad a \neq 0, \bar{0} \tag{3.31a}$$

$$\lim_{m \in 4Z \to \infty} [Y_m(0, 0, 1; q) + Y_m(\bar{0}, 0, 1; q)] = E(q^L, q^{2L})/\phi(q) \tag{3.31b}$$

$$\lim_{m \in 4Z \to \infty} [Y_m(0, 0, 1; q) - Y_m(\bar{0}, 0, 1; q)] = \phi(q)/\phi(q^2) \tag{3.31c}$$

(c) $D^{(1)}_{L+2}$ model:

$$\lim_{m \text{ even} \to \infty} Y_m(a, b, c; q) = \varepsilon_a^L E(-q^{a/2}, q^{L/2})/\phi(q) \tag{3.32}$$

(d) $A^{(1)}_{L-1}$ model. The case L is even:

$$\lim_{m \text{ even} \to \infty} Y_m(a, b, c; q) = E(-q^{a/2}, q^{L/2})/\phi(q) \tag{3.33a}$$

The case L is odd: The 1D configuration sum $Y_m(a, b, c; q)$ is nonzero in both cases: $a - b \equiv m$ and $a - b \not\equiv m \bmod 2$. Thus, we have

$$\lim_{m \text{ even} \to \infty} Y_m(a, b, c; q) = \frac{E(-q^{L/2+a}, q^{2L})}{\phi(q)} \qquad \text{if} \quad a \equiv b \bmod 2$$

$$= \frac{q^{a/2} E(-q^{L/2-a}, q^{2L})}{\phi(q)} \qquad \text{otherwise} \tag{3.33b}$$

Regime II. The 1D configuration sums in A_{L-1} and D_{L+1} models tend to the Hecke modular function $e^l_{j,k}(\tau)$ while those in $D^{(1)}_{L+2}$ and $A^{(1)}_{L-1}$ models become $c^{(+)}_{j,k}(q)$ in (A.10).

(a) A_{L-1} model. Among the admissible pairs (b, c) in the ground states (2.17) it is enough to consider the case $c = b + 1$ $(1 \leqslant b \leqslant L-2)$ due to the symmetry (3.9). Define $M(L; m, a, b)$ as

$$M(L; m, a, b) = \frac{m(m+1)}{4} - \frac{1}{4(L-2)} \left(m + \frac{L}{2} - b\right)^2 + \frac{1}{4L} \left(\frac{L}{2} - a\right)^2 + \frac{1}{24}. \tag{3.34}$$

Then we have

$$\lim_{\substack{m \to \infty \\ m \equiv \rho \bmod 2(L-2)}} q^{M(L;m,a,b)} X_m(a, b, b+1; q^{-1}) = e^{L-2}_{b-\rho-1,a-1}(\tau) \tag{3.35}$$

Here the parameter τ is related to q through $q = e^{2\pi i\tau}$ [see (1.7)] and $e^{l}_{j,k}(\tau)$ is Hecke's modular function described in Appendix B in ref. 7.

(b) D_{L+1} model. We give the results for the cases $(b, c) = (b, b+1)$, $1 \leqslant b \leqslant L-2$. The other cases are similar. By virtue of (3.11a) we assume that $a \neq \bar{0}$. Then we have

$$\lim_{\substack{m \to \infty \\ m \equiv \rho \bmod 2(2L-2)}} q^{M(2L;m,L+a,L+b)} X_m(a, b, b+1; q^{-1})$$

$$= \varepsilon_a^{\infty} [e^{2L-2}_{L+b-\rho-1,L+a-1}(\tau) + e^{2L-2}_{L+b-\rho-1,L-a-1}(\tau)] \tag{3.36}$$

Here $M(2L;\, m,\, L+a,\, L+b)$ is obtained from (3.34).

(c) $D^{(1)}_{L+2}$ model. The ground state (2.16), (2.23) has the structure of period 4. Due to the symmetries (3.13b), it is sufficient to consider the cases $(b, c) = (0, 1)$ and $(1, 0)$. In both cases we have

$$\lim_{m \in 4Z \to \infty} \tfrac{1}{2} [X_m(a, b, c; q^{-1}) + X_m(\bar{a}, b, c; q^{-1})]\, q^{m(m+1+c-b)/4}$$

$$= \varepsilon_a^L \frac{q^{(a-b)/2} E(-q^{L+a}, q^{2L})}{\phi(q)} \qquad \text{for} \quad a \neq \bar{0}, \bar{L} \tag{3.37a}$$

$$\lim_{m \in 4Z \to \infty} [X_m(0, 0, 1; q^{-1}) - X_m(\bar{0}, 0, 1; q^{-1})]\, q^{m^2/4}$$

$$= \phi(q)/\phi(q^2) \tag{3.37b}$$

(d) $A^{(1)}_{L-1}$ model. In view of (2.24) we give the results for $(b, c) = (0, L-1)$ and $(L-1, 0)$.

The case L is even:

$$\begin{aligned} &\lim_{m \text{ even} \to \infty} X_m(a, 0, L-1; q^{-1})\, q^{m^2/4} \\ &\quad = [q^{a/4} E(-q^{L-a}, q^{2L}) + q^{(L-a)/4} E(-q^a, q^{2L})]/\phi(q) \\ &\lim_{m \text{ even} \to \infty} X_m(a, L-1, 0; q^{-1})\, q^{m(m+2)/4} \\ &\quad = [q^{(L-1-a)/4} E(-q^a, q^{2L}) + q^{(a-1)/4} E(-q^{L-a}, q^{2L})]/\phi(q) \end{aligned} \tag{3.38a}$$

The case L is odd:

$$\begin{aligned}
&\lim_{m\text{ even}\to\infty} X_m(a,0,L-1;q^{-1})\,q^{m^2/4} \\
&\qquad = q^{a/4}E(-q^{L-a},q^{2L})/\phi(q) \qquad \text{if} \quad a \text{ is even} \\
&\qquad = q^{(L-a)/4}E(-q^{a},q^{2L})/\phi(q) \qquad \text{if} \quad a \text{ is odd} \\
&\lim_{m\text{ even}\to\infty} X_m(a,L-1,0;q^{-1})\,q^{m(m+2)/4} \\
&\qquad = q^{(L-1-a)/4}E(-q^{a},q^{2L})/\phi(q) \qquad \text{if} \quad a \text{ is even} \\
&\qquad = q^{(a-1)/4}E(-q^{L-a},q^{2L})/\phi(q) \qquad \text{if} \quad a \text{ is odd}
\end{aligned} \tag{3.38b}$$

Throughout regimes III and IV we use the variables r and s defined by (2.38).

Regime III

(a) A_{L-1} model:

$$\lim_{m\text{ even}\to\infty} X_m(a,b,c;q) = q^{(b-a)/4-\gamma(r,s,a)}c^{(-,+)}_{r,s,a}(q) \tag{3.39}$$

where $c^{(-,+)}_{r,s,a}(q)$ and $\gamma(r,s,a)$ are defined in (A.16) and (A.17), respectively, with m_1, m_2, m_3 taking the values (2.40).

(b) D_{L+1} model. Assuming (2.25) and $a\neq\bar{0}$, we have

$$\begin{aligned}
&\lim_{m\text{ even}\to\infty} X_m(a,b,c;q) \\
&\quad = \lim_{m\text{ even}\to\infty} X_m(\bar{a},b,c;q) \\
&\quad = \varepsilon^{\infty}_{a}q^{(b-a)/4-\gamma(L+r,s,L+a)}[c^{(-,+)}_{L+r,s,L+a}(q)+c^{(-,+)}_{L+r,s,L-a}(q)]
\end{aligned} \tag{3.40}$$

Here $c^{(-,+)}_{L+r,s,L\pm a}(q)$ and $\nu(L+r,s,L+a)$ are defined respectively in (A.16) and (A.17) with m_1, m_2, m_3 specified by (2.43).

(c) $D^{(1)}_{L+2}$ model. For a, b, $c\neq\bar{0}$, $\bar{L}$, we have

$$\begin{aligned}
&\lim_{m\text{ even}\to\infty} \tfrac{1}{2}[X_m(a,b,c;q)+X_m(\bar{a},b,c;q)] \\
&\qquad = \varepsilon^{L-1}_{r}q^{(b-a)/4-\gamma(r,s,a)}c^{(+,+)}_{r,s,a}(q)
\end{aligned} \tag{3.41a}$$

$$\lim_{m\text{ even}\to\infty} [X_m(0,0,1;q)-X_m(\bar{0},0,1;q)] = \phi(q)/\phi(q^2) \tag{3.41b}$$

where $c^{(+,+)}_{r,s,a}(q)$ and $\gamma(r, s, a)$ are defined in (A.16) and (A.17) with (2.40). Due to the symmetry (3.13b) the other cases are reduced to these cases.

(d) $A^{(1)}_{L-1}$ model. We assume (2.26) and obtain

$$\lim_{m \text{ even} \to \infty} \varepsilon_a^L X_m(a, b, c; q)$$
$$= \tfrac{1}{2} q^{(b-a)/4 - \gamma(r,s,a)} \{ c^{(+,+)}_{r,s,a}(q) + c^{(+,+)}_{r,s,L-a}(q) + \varepsilon_a^L [c^{(-,+)}_{r,s,a}(q) - c^{(-,+)}_{r,s,L-a}(q)] \} \tag{3.42}$$

where $\nu(r, s, a)$ and $c^{(\pm,+)}_{r,s,a}(q)$, etc., are defined in (A.16) and (A.17) with m_1, m_2, m_3 specified by (2.40).

Regime IV

(a) A_{L-1} model. Among the admissible pairs (b, c) satisfying (2.27) we deal with the case $b + c \leqslant 2n - 1$. The case $b + c \geqslant 2n + 3$ is reduced to this by (3.22).

The ease L is even:

$$\lim_{m \text{ even} \to \infty} \varepsilon_a^{L/2} [Y_m(a, b, c; q^{-1}) \mp Y_m(L-a, b, c; q^{-1})] q^{m(m+1+c-b)/4}$$
$$= q^{(b-a)/2 - \gamma(r,s,a)} c^{(-,\pm)}_{r,s,a}(q) \tag{3.43a}$$

The case L is odd:

$$\lim_{m \text{ even} \to \infty} Y_m(a, b, c; q^{-1})\, q^{m(m+1)/4 + m(c-b)/4}$$
$$= q^{(b-a)/2 - \gamma(r,s,a)} c^{(-,-)}_{r,s,a}(q) \tag{3.43b}$$

Here $c^{(-,\pm)}_{r,s,a}(q)$ and $\gamma(r,s,a)$ are defined in (A.16) and (A.17) with m_1, m_2, m_3 specified by (2.48).

(b) D_{L+1} model. For $a, b, c \neq \bar{0}$ we have

$$\tfrac{1}{2} \lim_{m \text{ even} \to \infty} [Y_m(a, b, c; q^{-1}) + Y_m(\bar{a}, b, c; q^{-1})] q^{m(m+1+b-c)/4}$$
$$= \varepsilon_r^{\infty} q^{-\gamma(r,s,a)} c^{(-,-)}_{L-1-r,3-s,L-a}(q) \tag{3.44a}$$

$$\lim_{m \text{ even} \to \infty} [Y_m(0, 0, 1; q^{-1}) - Y_m(\bar{0}, 0, 1; q^{-1})] q^{m^2/4} = \frac{\phi(q)}{\phi(q^2)} \tag{3.44b}$$

In (3.44a), $\gamma(r, s, a)$ and $c^{(-,-)}_{L-1-r,3-s,L-a}(q)$ are defined in (A.16) and (A.17) with m_1, m_2, m_3 taking the values (2.40). Due to the symmetry (3.24b) the other cases are reduced to the above.

(c) $D^{(1)}_{L+2}$ model. In view of the symmetry (3.25a), we assume that $a \neq \bar{0}, \bar{L}$. We deal with (b, c) satisfying the ground-state condition (2.28) and $b + c \leqslant 2n - 1$. The other cases are reduced to this by (3.22).

The case L is even:

$$\begin{aligned}
&\lim_{m \text{ even} \to \infty} \varepsilon^L_{a+L/2} Y_m(a, b, c; q^{-1}) q^{m(m+1+c-b)/4} \\
&\quad = \tfrac{1}{2} \varepsilon^L_a q^{(b-a)/2 - \gamma(r+L/2, s, a+L/2)} \\
&\qquad \times [c^{(+,+)}_{r+L/2, s, a+L/2}(q) + c^{(-,+)}_{r+L/2, s, a+L/2}(q) \\
&\qquad + c^{(+,+)}_{r+L/2, s, a-L/2}(q) - c^{(-,+)}_{r+L/2, s, a-L/2}(q)]
\end{aligned} \tag{3.45a}$$

The case L is odd:

$$\begin{aligned}
&\lim_{m \text{ even} \to \infty} Y_m(a, b, c; q^{-1}) q^{m(m+1+c-b)/4} \\
&\quad = \varepsilon^L_a q^{(b-a)/2 - \gamma(r+L/2, s, a+L/2)} \\
&\qquad \times [c^{(+,+)}_{r+L/2, s, a+L/2}(q) + c^{(+,+)}_{r+L/2, s, a-L/2}(q)]
\end{aligned} \tag{3.45b}$$

In (3.45) the power γ and the functions $c^{(\pm,+)}_{j_1 j_2 j_3}(q)$ are defined by (A.16) and (A.17) with m_1, m_2, m_3 specified by (2.40).

(d) $A^{(1)}_{L-1}$ model. We treat (b, c) that satisfies the ground-state condition (2.27) and $|b - c| = 1$, $b + c \leqslant 2n - 1$. The other cases are obtained by using (3.22) and (3.28).

The case L is even:

$$\begin{aligned}
&\lim_{m \text{ even} \to \infty} \varepsilon^L_{a+L/2} Y_m(a, b, c; q^{-1}) q^{m(m+1+c-b)/4} \\
&\quad = \tfrac{1}{2} q^{(b-a)/2 - \gamma(r+L/2, s, a+L/2)} \\
&\qquad \times [c^{(+,+)}_{r+L/2, s, a+L/2}(q) + c^{(-,+)}_{r+L/2, s, a+L/2}(q) \\
&\qquad + c^{(+,+)}_{r+L/2, s, a-L/2}(q) + c^{(-,+)}_{r+L/2, s, a-L/2}(q)]
\end{aligned} \tag{3.46a}$$

The case L is odd:

$$\begin{aligned}
&\lim_{m \text{ even} \to \infty} Y_m(a, b, c; q^{-1}) q^{m(m+1+c-b)/4} \\
&\quad = q^{(b-a)/2 - \gamma(r+L/2, s, a+L/2)} c^{(+,+)}_{r+L/2, s, a \pm L/2}(q) \\
&\qquad \text{for} \quad a \equiv b + \frac{1 \mp 1}{2} \bmod 2
\end{aligned} \tag{3.46b}$$

In (3.46) the power γ and the functions $c^{(\pm,+)}_{j_1 j_2 j_3}(q)$ are given by (A.16) and (A.17) with m_1, m_2, m_3 taking the values (2.40).

In Table V we summarize the functions $\eta(\tau)/\eta(2\tau)$, $c_{j,k}^{(\pm)}(q)$, $e_{j,k}^{l}(\tau)$, and $c_{j_1 j_2 j_3}^{(\pm,\pm)}(q)$ relevant to the $m\to\infty$ limit of the 1D configuration sums $X_m(a, b, c; q^{\pm 1})$ and $Y_m(a, b, c; q^{\pm 1})$.

Table V. The Branching Coefficients Appearing in the $m\to\infty$ Limit of the 1D Configurations Sums $X_m(a, b, c; q^{\pm 1})$ and $Y_m(a, b, c; q^{\pm 1})$[a]

Model			(m_1, m_2, m_3)
Regime I			
A_{L-1}		$c_{a,L}^{(-)}$	
D_{L+1}		$c_{L+a,2L}^{(-)}$, $\eta(\tau)/\eta(2\tau)$	
$D_{L+2}^{(1)}$		$c_{a/2,L/2}^{(+)}$	
$A_{L-1}^{(1)}$	L even	$c_{a/2,L/2}^{(+)}$	
	L odd	$c_{L/2\pm a,2L}^{(+)}$	
Regime II			
A_{L-1}		$e_{b-1,a-1}^{L-2}$	
D_{L+1}		$e_{L+b-1,L+a-1}^{2L-2}+e_{L+b-1,L-a-1}^{2L-2}$	
$D_{L+2}^{(1)}$		$c_{L+a,2L}^{(+)}$, $\eta(\tau)/\eta(2\tau)$	
$A_{L-1}^{(1)}$	L even	$c_{L-a,2L}^{(+)}+c_{a,2L}^{(+)}$	
	L odd	$c_{L-a,2L}^{(+)}$, $c_{a,2L}^{(+)}$	
Regime III			
A_{L-1}		$c_{r,s,a}^{(-,+)}$	$(L-1, 3, L)$
D_{L+1}		$c_{L+r,s,L+a}^{(-,+)}+c_{L+r,s,L-a}^{(-,+)}$	$(2L-1, 3, 2L)$
$D_{L+2}^{(1)}$		$c_{r,s,a}^{(+,+)}$, $\eta(\tau)/\eta(2\tau)$	$(L-1, 3, L)$
$A_{L-1}^{(1)}$		$c_{r,s,a}^{(+,+)}+c_{r,s,L-a}^{(+,+)}+c_{r,s,a}^{(-,+)}-c_{r,s,L-a}^{(-,+)}$	$(L-1, 3, L)$
Regime IV			
A_{L-1}	L even	$c_{r,s,a}^{(-,+)}+c_{r,s,a}^{(-,-)}$	$(L/2-1, 3, L/2)$
	L odd	$c_{r,s,a}^{(-,-)}$	$(L/2-1, 3, L/2)$
D_{L+1}		$c_{L-1-r,3-s,L-a}^{(-,-)}$, $\eta(\tau)/\eta(2\tau)$	$(L-1, 3, L)$
$D_{L+2}^{(1)}$	L even	$c_{r+L/2,s,a+L/2}^{(+,+)}+c_{r+L/2,s,a-L/2}^{(+,+)}$ $+c_{r+L/2,s,a+L/2}^{(-,+)}-c_{r+L/2,s,a-L/2}^{(-,+)}$	$(L-1, 3, L)$
	L odd	$c_{r+L/2,s,a+L/2}^{(+,+)}+c_{r+L/2,s,a-L/2}^{(+,+)}$	$(L-1, 3, L)$
$A_{L-1}^{(1)}$	L even	$c_{r+L/2,s,a+L/2}^{(+,+)}+c_{r+L/2,s,a-L/2}^{(+,+)}$ $+c_{r+L/2,s,a+L/2}^{(-,+)}+c_{r+L/2,s,a-L/2}^{(-,+)}$	$(L-1, 3, L)$
	L odd	$c_{r+L/2,s,a\pm L/2}^{(+,+)}$	$(L-1, 3, L)$

[a] r and s are integer parameters given in (2.38) and the function $c_{j_1 j_2 j_3}^{(\pm,\pm)}(q)$ is defined by (A.16) with (m_1, m_2, m_3) taking the specified values.

4. CRITICAL BEHAVIORS

The models A_{L-1}, D_{L+1}, $D^{(1)}_{L+2}$, and $A^{(1)}_{L-1}$ become critical as the elliptic nome p tends to zero, where the Boltzmann weights reduce to trigonometric functions. The aim of this section is to study the critical behaviors of the models and evaluate the exponents.

4.1. Free Energy

By the inversion method,[2,15] it is straightforward to compute the free energy per site. An explicit result for the A_{L-1} model is given by setting $N=1$ in (D.12)–(D.15) of ref. 8. Results for the other models are similar. Here we do not present the full expression, but focus on the relevant specific heat exponent α. It is extracted from the dominant singularity of the free energy (up to a factor $\log |p|$) as

$$f_{\text{sing}} \sim |p|^{2-\alpha} \tag{4.1}$$

From this we deduce the following values for the exponent α.

Regimes I and II. Here

$$\begin{aligned} 2-\alpha &= \frac{L}{L-2} \qquad \text{for} \quad A_{L-1}, D^{(1)}_{L+2}, \text{ and } A^{(1)}_{L-1} \text{ models} \\ &= \frac{L}{L-1} \qquad \text{for} \quad D_{L+1} \text{ model} \end{aligned} \tag{4.2a}$$

Regimes III and IV. Here

$$\begin{aligned} 2-\alpha &= L/2 \qquad \text{for} \quad A_{L-1}, D^{(1)}_{L+2}, \text{ and } A^{(1)}_{L-1} \text{ models} \\ &= L \qquad \text{for} \quad D_{L+1} \text{ model} \end{aligned} \tag{4.2b}$$

4.2. Local State Probabilities

In Section 2.3 we gave the LSP $P(a|A)$ in terms of the variable x, which goes to unity in the $p\to 0$ limit (Table I). (The LSP $P(a|A)$ should not be confused with the elliptic nome p.) Here we outline the way to rewrite them in suitable forms for studying the small-p behaviors and the method to get the exponents.

As stated in (2.29), the LSP $P(a|A)$ in general consists of the product of modular functions (or branching coefficients) $c_{a,A}(q)$ and ratios of specialized theta functions $T_{a,A}$ which are specified for each model and regime. The factor $T_{a,A}$ can be rewritten by applying so-called conjugate modulus identities for the theta functions $\Theta_{j,m}^{(\pm,\pm)}(x, x^2)$. In terms of the parameters p, ε, and x in Table I, these are given as follows.

A_{L-1} model in regimes I and IV:

$$x^{m/8}\Theta_{j,m}^{(-,+)}(x, x^2) = \left(\frac{\varepsilon}{\pi m}\right)^{1/2} \theta_1\left(\frac{\pi j}{m}, |p|^{2L/m}\right) \tag{4.3a}$$

$$x^{m/8}\Theta_{j,m}^{(+,+)}(x, x^2) = \left(\frac{\varepsilon}{\pi m}\right)^{1/2} \theta_4\left(\frac{\pi j}{m}, |p|^{2L/m}\right) \tag{4.3b}$$

$$x^{m/8}\Theta_{j,m}^{(-,-)}(x, x^2) = \left(\frac{\varepsilon}{2\pi m}\right)^{1/2} \theta_1\left(\frac{\pi j}{2m}, -|p|^{L/2m}\right) \tag{4.3c}$$

A_{L-1} model in regimes II and III and $D_{L+2}^{(1)}$, and $A_{L-1}^{(1)}$ models in all regimes:

$$x^{m/8}\Theta_{j,m}^{(-,+)}(x, x^2) = \left(\frac{\varepsilon}{2\pi m}\right)^{1/2} \theta_1\left(\frac{\pi j}{m}, p^{L/m}\right) \tag{4.3d}$$

$$x^{m/8}\Theta_{j,m}^{(+,+)}(x, x^2) = \left(\frac{\varepsilon}{2\pi m}\right)^{1/2} \theta_4\left(\frac{\pi j}{m}, p^{L/m}\right) \tag{4.3e}$$

For the D_{L+1} model all the necessary formulas are obtained from those for the A_{L-1} model by replacing L by $2L$ on the rhs of (4.3).

In the working below we use the variable t defined by

$$t = |p|^{2-\alpha} \tag{4.4}$$

with $2-\alpha$ given in (4.2). From Table I we see that t is related to the conjugate nome $\bar{q} = e^{-2\pi i/\tau}$ as follows:

$$\begin{aligned} \bar{q} &= t^2 \qquad \text{regimes I, IV of } A_{L-1}, D_{L+1} \text{ models} \\ &= t \qquad \text{otherwise} \end{aligned} \tag{4.5}$$

It turns out that as t tends to zero, $T_{a,\Lambda}$ vanishes as

$$T_{a,\Lambda} \propto t^{c/24} + \text{higher order terms in } t \tag{4.6}$$

Here c is a positive constant, shown in Table VI (the value c in ref. 7 for regime I of the A_{L-1} model should be corrected to $2-6/L$ for both parities of L).

For the branching coefficient $c_{a,\Lambda}(q)$, we apply the transformation formulas (A.12) and (A.20), etc., under the change $q = e^{2\pi i\tau} \to \bar{q} = e^{-2\pi i/\tau}$. Together with the relation (4.5), this yields the small-t expansion of the $c_{a,\Lambda}(q)$ as the linear combination of

$$t^{-c/24+\Delta}[1 + O(t)], \qquad \Delta \geqslant 0 \tag{4.7}$$

Table VI. The Values of c in (4.6) and (4.7) for Each Model and Regime

	I	II	III	IV
A_{L-1}	$2(1-3/L)$	$2(1-3/L)$	$1-6/L(L-1)$	$2[1-6/L(L-2)]$
D_{L+1}	$2(1-3/2L)$	$2(1-3/2L)$	$1-3/L(2L-1)$	$2[1-3/2L(L-1)]$
$D^{(1)}_{L+2}$	1	1	1	1
$A^{(1)}_{L-1}$	1	1	1	1

with c taking the values in Table VI. Thus, in the limit $t \to 0$, the LSP $P(a \mid \Lambda) = c_{a,\Lambda}(q) T_{a,\Lambda}$ converges to a finite value $P_a^{(c)}$ corresponding to the contributions having $\Lambda = 0$ in (4.7). In fact, the $P_a^{(c)}$ is independent of regimes and the boundary states, showing that the system has no long-range order at criticality. Below we list the values of $P_a^{(c)}$. By virtue of the symmetry $P_a^{(c)} = P_{\bar{a}}^{(c)}$, we assume that $a \neq \bar{0}$ (resp. $a \neq \bar{0}$, $\bar{L}$) for the D_{L+1} (reps. $D^{(1)}_{L+2}$) model. We have

$$
\begin{aligned}
P_a^{(c)} &= \frac{4}{L} \sin^2\left(\frac{\pi a}{L}\right) && A_{L-1} \text{ model} \\
&= \frac{4}{L}\left[\varepsilon_a^{\infty} \cos\left(\frac{\pi a}{L}\right)\right]^2 && D_{L+1} \text{ model} \\
&= \frac{2}{L}(\varepsilon_a^L)^2 && D^{(1)}_{L+2} \text{ model} \qquad (4.8) \\
&= \frac{2}{L} && A^{(1)}_{L-1} \text{ model, even } L \\
&= \frac{1}{L} && A^{(1)}_{L-1} \text{ model, odd } L
\end{aligned}
$$

We remark that these values coincide with the squared components of properly normalized eigenvector $\mathbf{h}$ for the matrix C in (1.16b) with the eigenvalue $q^{1/2}$ given by (1.23).

In view of (4.4), we consider that the minimum positive power Λ appearing in (4.7) is related to the exponent β through the relation [provided that the higher order terms in (4.6) do not contribute]

$$\Lambda = \beta/(2-\alpha) \tag{4.9}$$

Note that the standard scaling hypothesis asserts that $\eta = 4\Lambda$, where η is the "anomalous dimension."

Among the models and regimes, there are cases in which the results admit Lie algebraic interpretations. In such cases, the condition $1 = \sum_{a \in S} c_{a,\Delta}(q) T_{a,\Delta}$ is taken as the character identity (divided by the lhs) describing the irreducible decomposition of affine Lie algebra modules. As a consequence, $c_{a,\Delta}(q)$ turns out to be the (not necessarily irreducible) character of GKO-Virasoro algebra[22]

$$\mathrm{Vir} = \bigoplus_n CK_n \oplus Cc$$

with c the central charge and Δ constituting the spectrum of K_0. See refs. 7 and 10 for this "corresponding principle" between LSPs and the irreducible decomposition of characters for affine Lie algebras. Table VII list such cases and gives the relevant Lie algebra pairs with the level of their representations. (In regime II, another choice for the GKO pair is possible thanks to the duality between rank and level. See ref. 10.) In these cases the values of c and Δ realize those in conformal field theories (CFTs). The models A_{L-1} and D_{L+1} in regime III correspond to minimal theory,[5] while in regime II they are related to Z_{L-2}- and Z_{2L-2}-symmetric CFTs,[23] respectively. As noted in Section 1.3, the local states and their adjacent conditions for the $D_5^{(1)}$ model are equivalent to the odd-height sector of the fusion model[6-8] with $(L, N) = (6, 2)$. Actually, in regime III they share the same LSPs and correspond to $N = 1$ supersymmetric CFT[24] having $c = 1$. In regime II the 1D configuration sum of the $D_5^{(1)}$ model is related to the Hecke modular function $e^4_{j,k}(\tau)$ arising from the pair $(A_7^{(1)}, C_4^{(1)})$. However, the character identity describing the relation $A_7^{(1)} \supset C_4^{(1)}$ does not give the "sums-of-products" identity[3] necessary for the calculation of the LSP.

Table VII. The LSP Results That Admit Lie Algebraic Interpretations[a]

	Regime II	Regime III
A_{L-1} model	$(A^{(1)}_{2L-5}, C^{(1)}_{L-2})$	$(A_1^{(1)} \oplus A_1^{(1)}, A_1^{(1)})$
Level	1 1	$L-3$ 1 $L-2$
D_{L+1} model	$(A^{(1)}_{4L-5}, C^{(1)}_{2L-2})$	$(A_1^{(1)} \oplus A_1^{(1)}, A_1^{(1)})$
Level	1 1	$2L-3$ 1 $2L-2$
$D_5^{(1)}$ model	—	$(A_1^{(1)} \oplus A_1^{(1)}, A_1^{(1)})$
Level	—	2 2 4

[a] Relevant affine Lie algebra pairs (so-called GKO pairs) are listed with the level of their representations.

4.3. Spectrum of the Power Δ

Here we consider regimes II (A_{L-1}, D_{L+1} models), III, and IV, where the LSP $P(a|\Lambda)$ has a nontrivial dependence on the background configuration Λ as well as the central state a. We present the small-t expansion of relevant branching coefficients in the form (4.7). This reveals "fine structure" in the spectrum of the Δ in (4.7). In what follows the variable t is defined by (4.4) and (4.5) and the constant c is specified in Table VI.

Regime II (Table V). Here we assume that $a \equiv b \bmod 2$.

(a) A_{L-1} model ($c = 2 - 6/L$). The transformation formula and small-q behavior of $e^{l}_{j,k}(\tau)$ in Eqs. (B.7) and (B.8) in ref. 7 yield the following:

$$e^{L-2}_{b-1,a-1}(\tau) = \sum_{\substack{0 \leqslant j \leqslant k \leqslant L-2 \\ j \equiv k \bmod 2}} \frac{4}{[L(L-2)]^{1/2}} \varepsilon^{L-2}_{j} \varepsilon^{\infty}_{j-k}$$

$$\times \left(\cos \frac{\pi j(b-1)}{L-2} \sin \frac{\pi(k+1)a}{L} \right) t^{-c/24 + \Delta(j,k)} (1 + \cdots) \tag{4.10a}$$

$$\Delta(j,k) = -\frac{j^2}{4(L-2)} + \frac{k(k+2)}{4L} \tag{4.10b}$$

where the symbol ε^{l}_{j} is defined in (1.8).

(b) D_{L+1} model ($c = 2 - 3/L$):

$$e^{2L-2}_{L+b-1,L+a-1}(\tau) + e^{2L-2}_{L+b-1,L-a-1}(\tau)$$

$$= \sum_{0 \leqslant j \leqslant k \leqslant L-1} \frac{4}{[L(L-1)]^{1/2}} \varepsilon^{L-1}_{j} \varepsilon^{\infty}_{j-k}$$

$$\times \left(\cos \frac{\pi j b}{L-1} \cos \frac{\pi(2k+1)a}{2L} \right)$$

$$\times t^{-c/24 + \Delta(j,k)} (1 + \cdots) \tag{4.11a}$$

$$\Delta(j,k) = \frac{-j^2}{2(L-1)} + \frac{k(k+1)}{2L} \tag{4.11b}$$

Regime III (Table V). We assume that $r+s \equiv a+1 \bmod 2$.

(a) A_{L-1} model $[c=1-6/L(L-1)]$

$$c^{(-,+)}_{r,s,a}(q)=\frac{4}{[2L(L-1)]^{1/2}}\,\phi(t)^{-1} \times \sum_{n=1}^{\infty}\left(\sin\frac{\pi nr}{L-1}\sin\frac{\pi na}{L}\right)t^{-c/24+(n^2-1)/4L(L-1)} \tag{4.12}$$

(b) D_{L+1} model $[c=1-3/L(2L-1)]$:

$$\begin{aligned}&c^{(-,+)}_{L+r,s,L+a}(q)+c^{(-,+)}_{L+r,s,L-a}(q)\\&=\frac{4}{[L(2L-1)]^{1/2}}\,\phi(t)^{-1}\sum_{n=1}^{\infty}(-)^{n-1}\\&\left(\sin\frac{\pi(2n-1)(L+r)}{2L-1}\cos\frac{\pi(2n-1)a}{2L}\right)\\&\times t^{-c/24+n(n-1)/2L(2L-1)}\end{aligned} \tag{4.13}$$

(c) $D^{(1)}_{L+2}$ model $(c=1)$

$$c^{(+,+)}_{r,s,a}(q)=\frac{2\varepsilon_a^L}{[2L(L-1)]^{1/2}}\,t^{-c/24}\phi(t)^{-1} \times\left[1+2\sum_{n=1}^{\infty}\left(\cos\frac{\pi nr}{L-1}\cos\frac{\pi na}{L}\right)t^{n^2/4L(L-1)}\right] \tag{4.14a}$$

Besides the branching coefficient $c^{(+,+)}_{r,s,a}(q)$, the difference of the LSPs $P(0\,|\,0,1)-P(\bar{0}\,|\,0,1)$ in (2.44b) and (2.44c) behaves for small t as [Eq. (4.6) in ref. 9 should be corrected in this way]

$$\begin{aligned}&P(0\,|\,0,1)-P(\bar{0}\,|\,0,1)\\&=2\left(\frac{L-1}{L}\right)^{1/2} t^{1/16}\,\frac{\phi(t^2)\,\phi(t^{1/L})^2\,\phi(t^{2/(L-1)})}{\phi(t^{1/2})\,\phi(t^{1/(L-1)})^2\,\phi(t^{2/L})}\end{aligned} \tag{4.14b}$$

(d) $A^{(1)}_{L-1}$ model $(c=1)$. The case L is even:

$$\begin{aligned}&c^{(+,+)}_{r,s,a}(q)+c^{(-,+)}_{r,s,a}(q)+c^{(+,+)}_{r,s,L-a}(q)-c^{(-,+)}_{r,s,L-a}(q)\\&=\frac{4\varepsilon_a^L}{[2L(L-1)]^{1/2}}\,t^{-c/24}\\&\quad\times\phi(t)^{-1}\left\{1+2\sum_{n=1}^{\infty}\cos\left[2\pi n\frac{Lr-(L-1)a}{L(L-1)}\right]t^{n^2/L(L-1)}\right.\end{aligned} \tag{4.15}$$

The case L is odd: Under the assumption $r+s \equiv a+1 \bmod 2$, only the first two terms on the lhs of (4.15) are nonzero. Thus we have

$$\begin{aligned} &c^{(+,+)}_{r,s,a}(q) + c^{(-,+)}_{r,s,a}(q) \\ &\quad = \frac{2\varepsilon_a^L}{[2L(L-1)]^{1/2}}\, t^{-c/24}\phi(t)^{-1} \\ &\qquad \times \left[1 + 2\sum_{n=1}^{\infty} \cos\left(\pi n \frac{Lr-(L-1)a}{L(L-1)}\right) t^{n^2/4L(L-1)}\right] \end{aligned} \tag{4.16}$$

Regime IV (Table V). As in regime III, we assume that $r+s \equiv a+1 \bmod 2$.

(a) A_{L-1} model $\{c = 2[1-6/L(L-2)]\}$

$$\begin{aligned} &(2\varepsilon_a^{L/2})^{-1}\,[c^{(-,+)}_{r,s,a}(q) + c^{(-,-)}_{r,s,a}(q)]\,|_{L:\text{even}} \\ &\quad = c^{(-,-)}_{r,s,a}(q)\,|_{L:\text{odd}} \\ &\quad = \frac{4}{[2L(L-2)]^{1/2}}\,\phi(t^2)^{-1} \\ &\qquad \times \sum_{n=1}^{\infty}\left(\sin\frac{n\pi r}{L-2}\sin\frac{\pi n a}{L}\right) t^{-c/24+(n^2-1)/2L(L-2)} \end{aligned} \tag{4.17}$$

(b) D_{L+1} model $\{c = 2[1-3/2L(L-1)]\}$

$$\begin{aligned} &c^{(-,-)}_{L-1-r,3-s,L-a}(q) \\ &\quad = \frac{4\varepsilon_a^L}{[2L(L-1)]^{1/2}}\,\phi(t^2)^{-1}\sum_{n=1}^{\infty}\left(\cos\frac{\pi(2n-1)r}{2L-2}\cos\frac{\pi(2n-1)a}{2L}\right) \\ &\qquad \times t^{-c/24+n(n-1)/2L(L-1)} \end{aligned} \tag{4.18}$$

The difference of the LSPs $P(0\,|\,0,1) - P(\bar{0}\,|\,0,1)$ of (2.50b) is expressed in terms of t as

$$\begin{aligned} &P(0\,|\,0,1) - P(\bar{0}\,|\,0,1) \\ &\quad = 2\left(\frac{L-1}{L}\right)^{1/2} t^{(L^2-L-1)/8L(L-1)} \\ &\qquad \times \frac{\phi(t^4)\,\phi(t^{2/(L-1)})\,\phi(t^{1/L})\,\phi(t^{4/L})}{\phi(t)\,\phi(t^{1/(L-1)})\,\phi(t^{4/(L-1)})\,\phi(t^{2/L})} \end{aligned} \tag{4.19}$$

(c) $D^{(1)}_{L+2}$ model ($c=1$):

$$(2\varepsilon^{L}_{a+L/2})^{-1}\,[c^{(+,+)}_{r+L/2,s,a+L/2}(q)+c^{(+,+)}_{r+L/2,s,a-L/2}(q)$$
$$+c^{(-,+)}_{r+L/2,s,a+L/2}(q)-c^{(-,+)}_{r+L/2,s,a-L/2}(q)]\,|_{L:\text{even}}$$
$$=c^{(+,+)}_{r+L/2,s,a+L/2}(q)+c^{(+,+)}_{r+L/2,s,a-L/2}(q)\,|_{L:\text{odd}}$$
$$=\frac{2}{[2L(L-1)]^{1/2}}\,t^{-c/24}\phi(t)^{-1}$$
$$\times\left[1+2\sum_{n=1}^{\infty}\left(\cos\frac{\pi n(r+1/2)}{L-1}\cos\frac{\pi na}{L}\right)t^{n^2/4L(L-1)}\right] \tag{4.20}$$

(d) $A^{(1)}_{L-1}$ model ($c=1$). The case L is even:

$$(4\varepsilon^{L}_{a+L/2})^{-1}\,[c^{(+,+)}_{r+L/2,s,a+L/2}(q)+c^{(+,+)}_{r+L/2,s,a-L/2}(q)$$
$$+c^{(-,+)}_{r+L/2,s,a+L/2}(q)+c^{(-,+)}_{r+L/2,s,a-L/2}(q)]$$
$$=\frac{1}{[2L(L-1)]^{1/2}}\,t^{-c/24}\phi(t)^{-1}$$
$$\times\left[1+2\sum_{n=1}^{\infty}\cos\left(2\pi n\frac{L(r+1/2)-(L-1)a}{L(L-1)}\right)t^{n^2/L(L-1)}\right] \tag{4.21}$$

The case L is odd:

$$c^{(+,+)}_{r+L/2,s,a\pm L/2}(q)$$
$$=\frac{1}{[2L(L-1)^{1/2}}\,t^{-c/24}\phi(t)^{-1}$$
$$\times\left[1+2\sum_{n=1}^{\infty}\cos\left(\pi n\frac{L(r+1/2)\mp(L-1)a}{L(L-1)}\right)t^{n^2/4L(L-1)}\right] \tag{4.22}$$

4.4. Disordered Regimes

Here we consider the remaining regimes I and II ($D^{(1)}_{L+2}$ and $A^{(1)}_{L-1}$ models), where the LSPs have a trivial dependence on the boundary states. We give their explicit expressions in terms of the variable t, from which the exponent Δ in (4.9) is readily seen.

Regime I

(a) A_{L-1} model [Eqs. (2.30a) and (2.30b)]:

$$P(a)=\frac{2}{L}\,\frac{\theta_1(\pi a/L,\,t^{2/L})\,\theta_1(\pi a/L,\,-t^{(L-2)/L})}{\theta_1(\pi/2,\,-t)\,\theta_4(\pi\mu/2,\,t^{2(L-2)})} \tag{4.23}$$

where $\mu=0$ or 1 according as L is even or odd, respectively.

(b) D_{L+1} model [Eqs. (2.31a)–(2.31e)]. The LSP $P(a)$ [(2.31a), (2.31b)] and the difference $P(0|0,1)-P(\bar{0}|0,1)$ [(2.31d)] are rewritten as

$$P(a)=\frac{(\varepsilon_a^\infty)^2}{L}\frac{\theta_1(\pi(L+a)/2L,\ t^{1/L})\,\theta_1(\pi(L+a)/2L,\ -t^{(L-1)/L})}{\theta_1(\pi/2,\ -t)\,\theta_4(0,\ t^{4(L-1)})} \tag{4.24a}$$

$$\begin{aligned}&P(0|0,1)-P(\bar{0}|0,1)\\&\quad=\frac{1}{\sqrt{L}}\,t^{(L-1)/8L}\frac{\phi(t^2)^3\,\phi(t^{(L-1)/L})\,\phi(t^{4(L-1)/L})\,\phi(t^{4(L-1)})}{\phi(t)^2\,\phi(t^4)\,\phi(t^{2(L-1)/L})\,\phi(t^{2(L-1)})^2}\end{aligned} \tag{4.24b}$$

(c) $D_{L+2}^{(1)}$ model [Eqs. (2.32a), (2.32b)]:

$$P(a)=\frac{2}{L}(\varepsilon_a^L)^2\frac{\theta_4(\pi(L+2a)/2L,\ t^{(L-2)/2L})\,\theta_4(\pi a/L,\ t^{1/L})}{\theta_4(\pi(L-2\mu)/4,\ t^{(L-2)/4})\,\theta_4(\pi,\ t^{1/2})} \tag{4.25}$$

where μ is 0 or 1 according as a is even or odd, respectively.

(d) $A_{L-1}^{(1)}$ model [Eqs. (2.33a), (2.33b)]. When L is even, the LSP result (2.33a) is identical with that for the $D_{L+2}^{(1)}$ model, Eq. (2.32), up to a factor $(\varepsilon_a^L)^2$. Here we rewrite the odd L result (2.33b):

$$P^{(\pm)}(a)=\frac{1}{L}\frac{\theta_4(\pi(L\pm 2a)/2L,\ t^{(L-2)/L})\,\theta_4(\pi(L\pm 2a)/4,\ t^{1/2L})}{\theta_4(\pi(L-2\mu)/4,\ t^{(L-2)/2})\,\theta_4(\pi,\ t)} \tag{4.26}$$

where μ is 0 or 1 according as $a+(1\mp 1)/2$ is even or odd, respectively.

Regime II

(a) $D_{L+2}^{(1)}$ model [Eq. (2.36a)–(2.36e)]. The LSP $P(a)$ [Eqs. (2.36a), (2.36b)] and the difference $P(0|0,1)-P(\bar{0}|0,1)$ [Eq. (2.36d)] are expressed in terms of the variable t as follows:

$$P(a)=\frac{2}{L}(\varepsilon_a^L)^2\frac{\theta_4(\pi a/L,\ t^{(L-2)/L})\,\theta_4(\pi(L+a)/2L,\ t^{1/2L})}{\theta_4(\pi\mu/2,\ t^{(L-2)/2})\,\theta_4(\pi,\ t)} \tag{4.27a}$$

$$\begin{aligned}&P(0|0,1)-P(\bar{0}|0,1)\\&\quad=\frac{2}{\sqrt{L}}\,t^{1/16}\frac{\phi(t)^3\,\phi(t^{(L-2)/2})\,\phi(t^{(L-2)/2L})^2}{\phi(t^{1/2})^3\,\phi(t^{(L-2)/4})^2\,\phi(t^{(L-2)/L})}\end{aligned} \tag{4.27b}$$

Here μ in (4.27a) is 0 or 1 according as a is even or odd, respectively.

(b) $A^{(1)}_{L-1}$ model [Eqs. (2.37a), (2.37b)]. The LSP $P(a)$ in (2.37a) for even L and $P^{(\pm)}(a)$ in (2.37b) for odd L are written as

$$P(a)=\frac{2}{L}\frac{\theta_4(\pi a/L,\, t^{(L-2)/L})\,\theta_4(\pi a/L,\, t^{2/L})}{\theta_4(\pi\mu/2,\, t^{(L-2)/2})\,\theta_4(\pi,\, t)} \tag{4.28a}$$

$$P^{(\pm)}(a)=\frac{1}{L}\frac{\theta_4(\pi a/L,\, t^{(L-2)/L})\,\theta_4(\pi(L/2\pm(a-L/2))/2L,\, t^{1/2L})}{\theta_4(\pi\mu/2,\, t^{(L-2)/2})\,\theta_4(\pi,\, t)} \tag{4.28b}$$

where μ is 0 or 1 corresponding to a even or odd, respectively.

5. CONCLUDING REMARKS

In this paper we have exactly computed the local state probabilities (LSPs) for the four sequences of solvable RSOS models A_{L-1} $(L\geqslant 4)$, D_{L+1} $(L\geqslant 3)$, $D^{(1)}_{L+2}$ $(L\geqslant 3)$, and $A^{(1)}_{L-1}$ $(L\geqslant 3)$. They are characterized by the corresponding Dynkin diagrams in Figs. 1–4, showing the admissibility conditions for local state pairs to occupy adjacent lattice sites and the elliptic parametrization of the Boltzmann weights. The results include the previous work for the A_{L-1} model,[3] the $D^{(1)}_{L+2}$ model in regime III,[9] and the D_{L+1} model in regime III.[11] In all four regimes I–IV, the LSPs are evaluated in the form $P(a\,|\,\Lambda)=c_{a,\Lambda}(q)\,T_{a,\Lambda}$ [(2.29)] with $T_{a,\Lambda}$ a ratio of specialized theta function and $c_{a,\Lambda}(q)$ expressed by branching coefficients and linear combinations thereof. They obey definite transformation formulas under the change $q=e^{2\pi i\tau}\rightarrow\tilde{q}=e^{-2\pi i/\tau}$. Physically, this corresponds to the interchange of the vicinity of the critical point $(p=0, q=1)$ and extreme order/disorder $(|p|=1,\ q=0)$. By using such properties, we have studied the critical behaviors of the LSPs and the $c_{a,\Lambda}(q)$.

Here we remark that a solvable RSOS model is not uniquely specified by the set of local states and their neighboring conditions. Actually, Jimbo *et al.*[14] recently constructed yet other elliptic solutions to the STR for the RSOS model corresponding to the diagram in Fig. 3. One must go into the detailed structures in the Boltzmann weight parametrization in order to fully characterize such "exotic" models.

Through our analysis, we have found that the D_{L+1} model looks like the A_{2L-1} model. They manifest significant differences in the fractional powers Δ occurring in the small-t expansion of the relevant branching coefficients. Actually, compared with the A_{2L-1} model, there are some absences in the spectrum of Δ for the D_{L+1} model, as observed in Section 4.3. Such subtle structure seem analogous to the "operator content" of minimal conformal field theory on a torus.[25] There one deals with sesquilinear forms in the Virasoro character (1.2) and is led to the A–D–E classification[26] by the postulate of modular invariance. From this point of view[13] it would be interesting to seek an elliptic (off-critical) extensions of the RSOS models that correspond to exceptional cases: E_6, E_7, and E_8.

In many respects, the $D^{(1)}_{L+2}$ and $A^{(1)}_{L-1}$ models turn out to be distinct from the A_{L-1} and D_{L+1} models. This is most apparent in the values of $q^{1/2}$ and c given in (1.23a)–(1.23c) and Table VI, respectively. [The parameter q introduced in (1.14b) should not be confused with the argument $q=e^{2\pi i\tau}$ of modular functions.] Both of them depend on L for the A_{L-1} and D_{L+1} models, whereas $q^{1/2}=2$ and $c=1$ for the $D^{(1)}_{L+2}$ and $A^{(1)}_{L-1}$ models for all L. The reason for the difference of $q^{1/2}$ comes from (1.16b) and (1.20), which assert in the present cases that $2-q^{1/2}$ is the smallest eigenvalue of Cartan matrices for the classical Lie algebras A_{L-1} and D_{L+1} or affine Lie algebras $D^{(1)}_{L+2}$ and $A^{(1)}_{L-1}$. The constant c introduced in (4.6) and (4.7) in relation to the branching coefficients is the central charge of the Virasoro algebra for the cases listed in Table VI. We see that the values $c=1$ and $q=4$ for the $D^{(1)}_{L+2}$ and $A^{(1)}_{L-1}$ models consistently arise in regime III as the formal $g\to\infty$ limit of $c=1-6/g(g-1)$ and $q=4\cos^2(\pi/g)$ obtained for the A_{L-1} ($g=L$; Coxeter number) and D_{L+1} ($g=2L$) models. Just as the value $c=1-6/L(L-1)$ ($L=4, 5, 6, \ldots$) has a special meaning for the Virasoro algebra, so does the value $q=4\cos^2(\pi/L)$ for the Temperley–Lieb algebra appearing in II_1 factors.[27] Such a correspondence between the spectrum of the central charge of the Virasoro algebra and the index for subfactors for II_1 factors was first observed in ref. 18 for the A_{L-1} (restricted 8VSOS) model and is now extended to the correspondence between[10] $c=(n-1)[1-n(n+1)/L(L-1)]$ and $q=\sin^2(n\pi/L)/\sin^2(\pi/L)$.[28] All these follow from the miracles and mysteries of solvable lattice models, whose LSP involves the Virasoro characters away from criticality and whose STR yields the Hecke algebra representations of the braid groups at criticality.

APPENDIX. THETA FUNCTION IDENTITIES

In this appendix we introduce the elliptic theta function $\Theta^{(\pm,\pm)}_{j,m}(z, q)$ and describe the basic properties of the branching coefficients $c^{(\pm)}_{j,l}(q)$ and $c^{(\pm,\pm)}_{j_1 j_2 j_3}(q)$ appearing in appropriate identities among them. For the Hecke modular function $e^{l}_{j,k}(\tau)$ relevant to regime II of A_{L-1} and D_{L+1} models, see Appendix B in ref. 7.

A.1. The Theta Function $\Theta^{(\varepsilon_1,\varepsilon_2)}_{j,m}(z, q)$

For ε_1, $\varepsilon_2=\pm 1$, j, $m\in Z/2$, and $m>0$, we define an elliptic theta function $\Theta^{(\varepsilon_1,\varepsilon_2)}_{j,m}(z, q)$ by

$$\Theta^{(\varepsilon_1,\varepsilon_2)}_{j,m}(z, q)=\sum_{\nu\in Z,\gamma=\nu+j/2m}(\varepsilon_2)^{\nu}\,q^{m\gamma^2}(z^{-m\gamma}+\varepsilon_1 z^{m\gamma}) \tag{A.1}$$

It has the following symmetry and quasiperiodicity:

$$\begin{aligned}\Theta_{j,m}^{(\varepsilon_1,\varepsilon_2)}(z,q)&=\varepsilon_1\Theta_{-j,m}^{(\varepsilon_1,\varepsilon_2)}(z,q)\\&=\varepsilon_2\Theta_{j+2m,m}^{(\varepsilon_1,\varepsilon_2)}(z,q)\end{aligned}\tag{A.2}$$

$$\begin{aligned}\Theta_{j,m}^{(\varepsilon_1,\varepsilon_2)}(z,q)&=\varepsilon_1\Theta_{j,m}^{(\varepsilon_1,\varepsilon_2)}(z^{-1},q)\\&=\varepsilon_2(zq)^m\,\Theta_{j,m}^{(\varepsilon_1,\varepsilon_2)}(zq^2,q)\end{aligned}\tag{A.3}$$

$$\Theta_{j,m}^{(\varepsilon_1,\varepsilon_2)}(z,q)=\varepsilon_1(z^{-2}q)^{m/4}\,\Theta_{m-j,m}^{(\varepsilon_1\varepsilon_2,\varepsilon_2)}(zq^{-1},q)\tag{A.4}$$

$$\Theta_{j,m}^{(\varepsilon_1,\varepsilon_2)}(z,q)=(-)^j\,\Theta_{j,m}^{(\varepsilon_1,\varepsilon_2)}(ze^{2\pi i},q)\qquad\text{if}\quad j,m\in Z\tag{A.5}$$

Under the change

$$(z,q)=(e^{2\pi iu},e^{2\pi i\tau})\to(\bar z,\bar q)=(e^{2\pi iu/\tau},e^{-2\pi i/\tau})\tag{A.6}$$

the theta functions transform as follows:

$$\begin{aligned}\Theta_{j,m}^{(\varepsilon_1,\varepsilon_2)}(\bar z,\bar q)=(-i\tau)^{1/2}\,e^{\pi imu^2/2\tau}&\sum_k^{(\varepsilon_1,(-)^{2j},\varepsilon_2,m)}\\&\times T_{j,k}^m(\varepsilon_1)\,\Theta_{k,m}^{(\varepsilon_1,(-)^{2j})}(z,q)\end{aligned}\tag{A.7a}$$

$$\begin{aligned}T_{j,k}^m(+)&=(2/m)^{1/2}\,\varepsilon_k^m\cos(\pi jk/m)\\T_{j,k}^m(-)&=-i(2/m)^{1/2}\,\varepsilon_k^m\sin(\pi jk/m)\end{aligned}\tag{A.7b}$$

where the symbol ε_k^m is given in (1.8). For $\sigma_l=\pm1$ $(l=1,2,3)$ and m $(>0)\in Z/2$, the summation $\sum_k^{(\sigma_1,\sigma_2,\sigma_3,m)}$ is defined to be over k satisfying the following conditions:

$$\text{(i)}\quad\begin{aligned}&0\leqslant k\leqslant m&&\text{if}\quad(\sigma_1,\sigma_2)=(+,+)\\&0<k<m&&\text{if}\quad(\sigma_1,\sigma_2)=(-,+)\\&0\leqslant k<m&&\text{if}\quad(\sigma_1,\sigma_2)=(+,-)\\&0<k\leqslant m&&\text{if}\quad(\sigma_1,\sigma_2)=(-,-)\end{aligned}\tag{A.8a}$$

$$\text{(ii)}\quad(-)^{2k}=\sigma_3\tag{A.8b}$$

The theta function becomes a simple infinite product under the specialization $z=x$, $q=x^2$:

$$x^{m/8}\Theta_{j,m}^{(\varepsilon_1,\varepsilon_2)}(x,x^2)=x^{(m-2j)^2/8m}\,E(-\varepsilon_1x^j,\varepsilon_2x^m)\tag{A.9}$$

where $E(z,q)$ is defined in (1.3). For fixed ε_1, ε_2, and m, theta functions $\{\Theta_{j,m}^{(\varepsilon_1,\varepsilon_2)}(z,q)\,|\,j\in J\}$ for an appropriate choice of $J\subset Z/2$ span the vector space that consists of functions having the same quasiperiodicity (A.3). In the sequel, we shall exploit this argument to define the branching coefficients.

A2. The Function $c_{j,l}^{(\varepsilon)}(q)$

For $\varepsilon = \pm 1$, $j \in Z/2$, $l\ (>0) \in Z$, we define a function $c_{j,l}^{(\varepsilon)}(q)$ to be a ratio of infinite products [see (1.7)]

$$c_{j,l}^{(\varepsilon)}(q) = \varepsilon c_{-j,l}^{(\varepsilon)}(q) = c_{l-j,l}^{(\varepsilon)}(q) = \frac{q^{(l-2j)^2/8l} E(-\varepsilon q^j, q^l)}{\eta(\tau)} \tag{A.10}$$

From the identity $E(z, q) = E(-z^2 q, q^4) - zE(-z^{-2} q, q^4)$, it immediately follows that

$$c_{j,l}^{(\varepsilon)}(q) = c_{l+2j,4l}^{(+)}(q) + \varepsilon c_{l-2j,4l}^{(+)}(q) \tag{A.11}$$

Through the change $q = e^{2\pi i \tau} \to \tilde{q} = e^{-2\pi i/\tau}$, following transformation formula is valid:

$$c_{j,l}^{(\pm)}(q) = \sum_{k}^{((-)^{l+2j},\, +,\, \pm(-)^l,\, l/2)} e^{(j+k-l/2)\pi i} \times T_{j,k}^{l/2}[\pm(-)^{2j}]\, c_{k,l}^{((-)^{l+2j})}(\tilde{q}) \tag{A.12}$$

where $T_{j,k}^{l/2}(\pm)$ is defined by (A.7b). The function $c_{j,l}^{(\varepsilon)}(q)$ for $j \in Z$, $l\ (>2)$ is characterized as a branching coefficient by a theta function identity of the form

$$\begin{aligned} &\Theta_{s,2}^{(+,+)}(z, x)\, \Theta_{l-2,l-2}^{(\varepsilon,\varepsilon)}(z, x) \\ &\quad = 2 \sum_{\substack{0 \leqslant j \leqslant l \\ j-1 \equiv (l+s)/2 \bmod 2}} \eta(\tau)\, \varepsilon_j^l c_{j,l}^{(\varepsilon)}(q) \\ &\quad\quad \times \Theta_{2j,l}^{(\varepsilon,\varepsilon)}(z, x), \qquad q = x^{l-2} \end{aligned} \tag{A.13}$$

where $s = 1$ if l is odd and $s = 0$ or 2 if l is even. The rhs gives an expansion of the lhs in terms of the basis having the same quasiperiodicity (A.3) with $(\varepsilon_1, \varepsilon_2, m) = (\varepsilon, \varepsilon, l)$.

Besides the identity (A.13), we also use the following identities involving the function $c_{j,L/2}^{(+)}(x^{L-2})$ reexpressed by (A.11):

$$\begin{aligned} &\sum_{\substack{a \in Z/2LZ \\ a \equiv v \bmod 2}} x^{a(a-L+2)/4} E(-x^{a+L/2}, x^L) \\ &\quad \times E(-x^{(L+2a)(L-2)/2}, x^{2L(L-2)}) \\ &\quad = x^{v(3-L)/4} E(-x^{L/2-v}, x^2)\, E(-x^{L-2}, x^{L-2}) \end{aligned} \tag{A.14}$$

$$\begin{aligned} &\sum_{\substack{a \in Z/2LZ \\ a \equiv v \bmod 2}} x^{a(a-2)/4} E(-x^a, x^L)\, E(-x^{(L+a)(L-2)}, x^{2L(L-2)}) \\ &\quad = x^{-v/4} E(-x^v, x^2)\, E(-x^{L-2}, x^{L-2}) \end{aligned} \tag{A.15}$$

where $\nu = 0$ or 1. The formula (A.14) [resp. (A.15)] can be derived by the method described in ref. 3 (pp. 238–239) ref. 3 if we replace Eq. (3.2.10) therein by

$$\sigma(a) = x^{a(a-1)/4} z^a E(-x^{a+L/2}, y), \qquad (y, z) = (x^L, x^{(3-L)/4})$$

$$[\text{resp. } \sigma(a) = x^{a(a-2)/4} z^a E(-x^a, y), \qquad (y, z) = (x^L, 1)].$$

A3. The Branching Coefficient $c^{(\varepsilon_1, \varepsilon_2)}_{j_1 j_2 j_3}(q)$

Assume that ε_1, $\varepsilon_2 = \pm 1$, j_1, $m_1 \in Z/2$, j_2, $m_2 \in Z$, $0 \leqslant j_i \leqslant m_i > 0$ $(i = 1, 2)$, and $m_3 = m_1 + m_2 - 2 > 0$. There exists a theta function identity of the form

$$\Theta^{(\varepsilon_1, \varepsilon_2)}_{j_1, m_1}(z, q) \frac{\Theta^{(-,+)}_{j_2, m_2}(z, q)}{\Theta^{(-,+)}_{1,2}(z, q)} = \sum_{j_3}^{(\varepsilon_1, \varepsilon_2, (-)^{2j_1}, m_3)} c^{(\varepsilon_1, \varepsilon_2)}_{j_1 j_2 j_3}(q)\, \Theta^{(\varepsilon_1, \varepsilon_2)}_{j_3, m_3}(z, q) \tag{A.16}$$

where the summation symbol has been specified in (A.8). As in (A.13), the rhs of (A.16) expresses the lhs as a linear combination of $\Theta^{(\varepsilon_1, \varepsilon_2)}_{j_3, m_3}(z, q)$ that enjoys the same quasiperiodicity (A.3) with $m = m_3$. This completely characterizes the entry $c^{(\varepsilon_1, \varepsilon_2)}_{j_1 j_2 j_3}(q)$ as the branching coefficient. When j_1, $m_1 \in Z$, the q-expansion of $c^{(\varepsilon_1, \varepsilon_2)}_{j_1 j_2 j_3}(q)$ contains a fractional power

$$\gamma(j_1, j_2, j_3) = \frac{j_1^2}{4m_1} + \frac{j_2^2}{4m_2} - \frac{1}{8} - \frac{j_3^2}{4m_3} \tag{A.17}$$

We remark that in the case $(\varepsilon_1, \varepsilon_2) = (-, +)$, j_1, $m_1 \in Z$, formula (A.16) divided by $\Theta^{(-,+)}_{1,2}(z, q)$ is the character identity describing the irreducible decomposition of tensor products of $A_1^{(1)}$ modules. As a consequence, the function $c^{(-,+)}_{j_1 j_2 j_3}(q)$ turns out to be the (not necessarily irreducible) character of the Virasoro algebra constructed from an affine Lie algebra pair $(A_1^{(1)} \oplus A_1^{(1)}, A_1^{(1)})$.

From (A.16) and (A.4)–(A.5) we deduce the following properties:

(i) $c^{(\varepsilon_1, \varepsilon_2)}_{j_1 j_2 j_3}(q) = c^{(\varepsilon_1 \varepsilon_2, \varepsilon_2)}_{m_1 - j_1, m_2 - j_2, m_3 - j_3}(q)$ (A.18a)

(ii) $c^{(\varepsilon_1, \varepsilon_2)}_{j_1 j_2 j_3}(q) = 0$ if j_1, $m_1 \in Z$ and $j_1 + j_2 \equiv j_3 \bmod 2$ (A.18b)

(iii) $c^{(\varepsilon, -)}_{j_1 j_2 j_3}(q) = (-)^{j_1 + j_2 + j_3 + 1} c^{(\varepsilon, +)}_{j_1 j_2 j_3}(q)$
if $m_1 \in Z + 1/2$ and $j_1 \in Z$ (A.18c)

In view of (A.2), we extend the definition of $c_{j_1 j_2 j_3}^{(\varepsilon_1, \varepsilon_2)}(q)$ as follows:

$$
\begin{aligned}
& c^{(\varepsilon_1, \varepsilon_2)}_{\sigma_1(j_1+2n_1m_1),\sigma_2(j_2+2n_2m_2),\sigma_3(j_3+2n_3m_3)}(q) \\
&\quad = \sigma_2(\varepsilon_1)^{1+(\sigma_1+\sigma_3)/2}\,(\varepsilon_2)^{n_1+n_3}\,c_{j_1 j_2 j_3}^{(\varepsilon_1, \varepsilon_2)}(q) \\
&\quad \text{for} \quad \sigma_i = \pm 1, \quad n_i \in Z, \quad i = 1, 2, 3
\end{aligned} \tag{A.19}
$$

The branching coefficient enjoys the following automorphic property as the direct consequence of that for the theta function (A.7):

$$
\begin{aligned}
c_{j_1 j_2 j_3}^{(\varepsilon_1, \varepsilon_2)}(q) = & \sum_{k_1}^{(\varepsilon_1, \rho, \varepsilon_2, m_1)} \sum_{k_2}^{(-,+,+,m_2)} \sum_{k_3}^{(\varepsilon_1, \rho, \varepsilon_2, m_3)} \\
& \times i\varepsilon_1\, T^{m_1}_{j_1, k_1}(\varepsilon_1)\, T^{m_2}_{j_2, k_2}(-)\, T^{m_3}_{k_3, j_3}(\varepsilon_1)\, c^{(\varepsilon_1, \rho)}_{k_1 k_2 k_3}(\bar{q})
\end{aligned} \tag{A.20}
$$

where $\rho = (-)^{2j_1} = (-)^{2j_3}$.

Finally we present the explicit expression of $c_{j_1 j_2 j_3}^{(\varepsilon_1, \varepsilon_2)}(q)$ for the case $(m_1, m_2, m_3) = (m-1, 3, m)$, which is relevant to our models. The case $\varepsilon_1 = -1$, $j_1 \in Z$, has been given in Appendix C of ref. 8. In what follows we assume the properties (A.18).

(i) $j_1, m \in Z$:

$$
c_{j_1 j_2 j_3}^{(\varepsilon_1, \varepsilon_2)}(q) = \varepsilon_{j_3}^{m}\eta(\tau)^{-1} \sum_{v \in Z} (\varepsilon_2)^v \left(q^{h^m_{j_1, j_3}(v)} + \varepsilon_1 q^{h^m_{j_1, -j_3}(v)}\right) \tag{A.21}
$$

(ii) $j_1 \in Z + 1/2$, $m \in Z$:

$$
\begin{aligned}
c_{j_1 j_2 j_3}^{(\varepsilon_1, \varepsilon_2)}(q) &= (\varepsilon_1)^{(1 \mp 1)/2}\,\eta(\tau)^{-1} \sum_{v \in Z} (\varepsilon_2)^v q^{h^m_{j_1, \pm j_3}(v)} \\
& \text{if} \quad \pm j_3 + 1 \equiv j_1 + j_2 \bmod 2
\end{aligned} \tag{A.22}
$$

(iii) $j_1 \in Z$, $m \in Z + 1/2$:

$$
\begin{aligned}
c_{j_1 j_2 j_3}^{(\varepsilon_1, \varepsilon_2)}(q) &= \varepsilon_1 \varepsilon_2 c^{(\varepsilon_1, \varepsilon_2)}_{j_1, j_2, 2m - j_3}(q) \\
&= \varepsilon_{j_3}^{2m}\eta(\tau)^{-1} \sum_{v \in Z} \left(q^{h^m_{j_1, j_3}(2v)} + \varepsilon_1 q^{h^m_{j_1, -j_3}(2v)}\right) \\
& \text{if} \quad j_3 + 1 \equiv j_1 + j_2 \bmod 2
\end{aligned} \tag{A.23}
$$

Here the symbol ε_j^l is specified in (1.8) and the power $h^m_{j,k}(v)$ is defined by

$$
h^m_{j,k}(v) = \frac{[2m(m-1)v + mj - (m-1)k]^2}{4m(m-1)} \tag{A.24}
$$

Due to the symmetry (A.18a), the case j_1, $m \in Z + 1/2$ is reduced to (A.23).

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